

## ELASTIC AND DYNAMIC RESPONSE REGIMES OF FLUID-IMPREGNATED SOLIDS WITH DIVERSE MICROSTRUCTURES

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(Received 19 August 1977; in revised form 27 February 1978; received for publication 14 April 1978)

**Abstract**—Three major regimes of response are identified for time-dependent deformation of a solid matrix which contains fluid globules free to communicate with each other and adjoining elements or atmosphere. The moduli pertaining to the four limits are estimated from specification of solid and fluid properties (moduli, concentrations, and geometry): alternative techniques are outlined but a rationalized version of the conventional self-consistent approximation is specifically developed and applied to special cases of interest, viz. spherical and flat ellipsoidal fluid inclusions. Characteristic times are identified for each regime which, along with frequent simultaneity of local inter-site and global inter-element diffusion, are emphasized as rendering suspect the classical and mixture theories for wave propagation: especially, the assumption of a single tensor-valued potential, to govern macroscopic fluid flow, has very limited application. Related complications for inelasticity formulations and “effective-stress” laws are discussed; special schemes are suggested to resolve the various difficulties but considerable further study is being undertaken.

### 1. INTRODUCTION

A variety of theories have accumulated over the past four decades on the related subjects of heterogeneous solids and fluid-saturated porous media. Methods for computing the effective moduli and other (e.g. thermal or electrical) properties of the former have been developed and improved, with occasional consideration of the case where the second phase is fluid-occupied porosity; central references in this respect are the treatments by Budiansky[1, 2] and Hill[3], the computations by Wu[4], Walsh[5] and others[6, 7], the more recent generalizations of Gubernatis and Krumhansl[8] and the various potentially useful crack-density tensor concepts[9]. Meanwhile, the basis for describing coupled fluid flow and solid matrix deformation has been developed and applied by Biot[10, 11, respectively], with recent rationalizations by Rice and Cleary[12, 13]. However, despite its frequent “extension” [e.g. 14–18] to describe the dynamic response of porous media infiltrated by one (or even many) fluid phases, the classical concept of a single fluid stress and flow vector (more recently cloaked in mixture theory formalism) has not been at all vindicated for the rapid stress alterations that many dynamic circumstances (e.g. geophysical wave propagation) would imply. A major goal in this paper is to lay bare the physical difficulties that may frequently obstruct such a simplified description of the medium.

In the first place, we employ concepts familiar from studies of composite solids to indicate that there are at least three distinct elastic responses that the fluid-saturated porous medium can exhibit. The first corresponds to *instantaneous* stiff shear response of the fluid (or other diffusive species located in distributed sites), the second derives from bulk stiffness but negligible shear modulus within each site (after viscous but *isolated relaxations*) and the third is distinguished by a negligible bulk or shear resistance of the various sites (after sufficient matter has been squeezed out that the stress-generated fluid pressure has been extinguished). The latter is referred to as the “*drained*” condition; rationalization[12] of Biot's theory then reveals *another* physically important elastic limit, which corresponds to the “*undrained*” case where no net flow of fluid is allowed out of the representative microstructural element under stress (although local exchange of fluid between sites is permitted). These four most obvious elastic limits imply at least three regimes of time-dependent response: the first is essentially *visco-elastic relaxation* within fluid sites (including, perhaps, the few realistic mechanisms proposed by Biot[15]). If the boundaries of the material element considered are sealed against fluid exchange with adjoining elements then the second regime is also a purely viscoelastic (but *diffusive*) *internal readjustment* of the microstructure; the result would be an equilibrated

potential of the fluid and subsequent removal of the jacketing (e.g. communication with adjacent stressed elements) would lead to a third regime of *global diffusion* viz. inter-element exchange of fluid until the potential reaches an equilibrium state throughout the stressed body. However, the sealing process is clearly an artifact which is not present in the real body: thus, if their characteristic times are similar, the *diffusive internal readjustment and global diffusion can be occurring simultaneously* and, therefore, *it may not be possible to define a single tensor-valued fluid potential which could be chosen to govern global diffusion*. Even if a single flow vector can be identified as representing the net mass flow of fluid in any direction, it is not obvious how a constitutive relation (analogous to that of D'Arcy) can be used to connect it with any fluid or solid stress measure. We do extract some possibilities in Section 4 but the question will merit substantially more consideration in the future.

We begin by providing the bases for estimates of the various limiting elastic moduli appearing in the description of the time-dependent regimes which porous media can exhibit. In particular, we focus on the self-consistent method of computation, so frequently used previously to extract the moduli of composites from a specification of moduli and layout (viz. volume concentrations and shapes) for the constituents. Our development has substantial novelty, for instance in the way that we treat the addition of inclusions (to achieve the current concentration) as an incremental process in which the energy change at each stage is computed on the basis of current moduli (at increasing fractions of the final concentration but with shapes and orientations exactly like the final geometry to be achieved). We treat the simple reference case of spherical inclusions and focus also on the other revealing extreme of flat ellipsoids: these allow us to develop transparent estimates for important effects of matrix inclusion moduli, exterior stress orientation and non-equiaxedness of pores.

Having established that moduli do change drastically, with alterations in the effective moduli of interior inclusions, we proceed (in Section 3) to compute typical times associated with each regime of response evolution. These are based simply on consideration of the primary mechanisms for relaxation (so that the shortest possible times are determined) and the general eigenvalue procedure is merely indicated in Section 4. A dominant conclusion is that times for inter-site and inter-element fluid exchange may frequently be found comparable in nature; exceptions are wavelengths so long that inappreciable global diffusion occurs anyway and we identify an obscured feature of classical theories, namely that wavelengths may actually have to be so short (for appreciable attenuation with reasonable wave velocities) that they become unacceptably comparable to microstructural dimensions.

## 2. ESTIMATION OF LIMITS FOR RESPONSE EVOLUTION

Our primary task is to establish that there are indeed drastic differences between the moduli which the composite representative structural element of material exhibits as the times allowed for measurement (and, often, the boundary-conditions) are varied. In standard viscoelasticity formulations (e.g. [19]) this transition through various intermediate regimes is rather generally attributed to internal rate processes but there is also a classical theory from soil mechanics [20] which provides a second major source of time-dependent compliance, namely the global diffusion of an internal species (such as pore-fluid) to adjoining "atmosphere" or adjacent elements. Both sources are possible in the present discussion and we wish to give specific character to some relaxation processes which are particularly relevant to fluid-saturated porous media (e.g. in the geophysical and geological context). We shall identify various operative mechanisms on the microstructural scale and we will need a number of approximate modelling methods to compute the consequences for the macroscopic response of a representative element.

### (a) *The self-consistent approximation*

It is often the case that a single well-defined feature dominates in the distinction between the real microstructure and that of a homogeneous single-phase material: for instance, the otherwise homogeneous matrix of primary phase may contain a distribution of second-phase particles, or even fluid-filled pores, which is adequately described as a repeated placement of a generic particle (or pore) with some limited order of orientation and size. If it is possible to focus on any generic site and honestly conclude that none of the neighboring structure has an

unavoidably discrete interaction with the chosen inhomogeneity (viz. one which cannot be “smeared out” and blended into the overall surrounding region), then a fairly obvious *modus operandi* suggests itself: this takes the form of a hypothesis that the *effect of any generic inhomogeneity site on the response of the overall structural element containing it can be sufficiently accurately computed by considering the site to be embedded in an otherwise homogeneous surrounding region subject to a uniform remote stress field*. Clearly, a number of variations on this computational artifact suggest themselves (e.g. gradients in the remote stress field on the scale of the site, correlation functions for site distribution [21] or even clustering of elementary inclusions to create a representative inhomogeneity); there is also an important ambiguity in choice of moduli for the exterior region—which we shall help to remove in this treatment. However, it is possible to classify the overall concept as the *self-consistent approximation*.†

A most appealing version of such a general procedure is one which computes how much a single inhomogeneity site perturbs the potential energy of the initially smooth stress field; by adding in an appropriate number of such sites per unit volume, in a fashion yet to be decided, the sum of perturbations in initial energy can be computed and thence the final energy per unit volume, resulting from nominally uniform stresses, can be estimated. Since that energy can also be expressed in terms of the stresses and moduli governing response of the composite element, the result is an estimate of the composite moduli. This approach is, indeed, “self-consistent” and the only major question is as to how, both in philosophy and detail, the computation for a single site is to be made and merged with the effect of all other sites. A quantum leap in progress on this question has been the provision of solutions for “the inclusion problem” (especially in [22, 23]‡), including the extreme case of a flat crack (more recently exploited, for general shapes in [6, 7]); for special but useful site idealizations, these allow us to compute the energy perturbations referred to above. What has not been so clearly established, and hence merits more careful discussion here, is which exterior moduli we should use in performing the estimates for a single site, since sufficient density of sites can change the effective moduli appreciably. If site concentration is dilute there is negligible computational ambiguity (e.g. as first treated in [5] for flat cracks), and if it is too great the basic assumption of the approximations becomes untenable (although the estimate may still not be bad in certain cases, e.g. twin solid phases); however, there is an intermediate level of site density (with accompanying moduli modifications by factors of 2 or more) where the question really does seem worthwhile pursuing, even from the viewpoint of physical reality (as indicated by some experimental results, e.g. [24]).

(i) *Solutions for ellipsoidal inclusions* (Fig. 1a). To clarify the procedure and identify some controversial questions, we shall schematically trace the methodology for estimating the moduli of a homogeneous isotropic solid matrix (shear modulus  $G^s$ , Poisson’s ratio  $\nu^s$ ), containing a distribution of ellipsoidal inclusions,§ which may be constituted of solid or fluid; shapes may include crack-like limits yet still strongly affect moduli, as we shall see. A central feature is the result of Eshelby [22] (expounded upon in [23]) that the *stress or strain field in such inclusions is actually uniform*. If we use  $\epsilon_{ij}^{inc}$  to denote that field and suppose it results from a remotely applied strain  $\epsilon_{ij}^A$  (or stress  $\sigma_{ij}^A \approx \tilde{L}_{ijkl}\epsilon_{kl}^A$  where  $\tilde{L}$  is the fourth-order isotropic¶ elasticity tensor of the surrounding matrix), then the remarkable result is most readily interpreted in the form

$$\epsilon_{ij}^{inc} = \epsilon_{ij}^c + \epsilon_{ij}^A, \quad \epsilon_{ij}^c \equiv S_{ijkl}\epsilon_{kl}^T, \quad (2.1)$$

†It is worth noting (e.g. see [21] for further discussion) that the first-order approximate solutions for exact integral formulations [8], which are claimed to be self-consistent, seem to coincide exactly with the hypothesis just outlined: indeed, such formulations demonstrate that many more sophisticated techniques may be generated to permit more exact descriptions by numerical computation. However, we are concerned here with schemes of which an available analytic solution (like that of Eshelby) can form the basis. Thus, if we knew the solution for adjacent pairs of inclusions (analogous to two spheres in Stokesian flow, as a model of settlement) we could probably improve our estimates: in such integral schemes, the corresponding improvement requires the rather tedious incorporation of second-order terms [e.g. 33].

‡In fact, we shall adhere to the relevant observations and notation of Eshelby, as closely as brevity will permit, in this paper: we do this for co-ordination with previous work and to avoid excess formalism (in which terminology has to be greatly distended anyway in preparation for quantitative estimates). When we merely wish to outline the computational scheme (e.g. [21] for the anisotropic counterparts of results presented here), we find the provably equivalent equations of Hill [3] more convenient as starting point.

§Fairly general results may be derived without specific reference to shapes of phase distributions (e.g. [3], eqn 2.4 of [1] and eqn 2.26 of [23]) but actual computations typically require such assumptions.

¶Corresponding (but less tractable) results are available for anisotropy (e.g. see [25]); their use is traced elsewhere [21].

where  $\epsilon_{ij}^T$  corresponds precisely to a transformation (plastic or misfit) strain of the kind which would be considered in trying to fit a dislocated ellipsoidal piece of material back into its original slot in an otherwise perfect matrix (this was the fundamental problem which led Eshelby to his most significant results). The *shape factors*  $S_{ijkl}$  arise in connection with such a misfit problem (and will be provided below); by a simple sequence of "cut and paste" operations for the inclusion, Eshelby [23, eqn (4.5)] deduced a fairly obvious implicit equation for  $\epsilon_{ij}^T$ , which we generalise as

$$(L^* - \tilde{L})_{ijmn} S_{mnlk} \epsilon_{kl}^T + \tilde{L}_{ijkl} \epsilon_{kl}^T = (\tilde{L} - L^*)_{ijkl} \epsilon_{kl}^A \quad (2.2)$$

where asterisk or tilde apply to the inclusion or matrix properties, respectively. As well as this, it is a straightforward matter (e.g. [23], eqns 4.13–4.15) to prove that the change in elastic potential energy of an extensive body, due to the introduction of a single inclusion of volume  $V^j$ , has the very simple form

$$\Delta E^j = -\frac{1}{2} \Delta V^j \sigma_{ik}^A \epsilon_{ik}^T, \quad V^j \equiv \frac{4\pi}{3} (a_1 a_2 a_3)^{ij} \quad (2.3)_1$$

where the isotropic specialisation reduces to

$$\sigma_{ij}^A \equiv \tilde{\lambda} \epsilon_{kk}^A \delta_{ij} + 2\tilde{G} \epsilon_{ij}^A, \quad \lambda \equiv 2G\nu/(1-2\nu) \quad (2.3)_2$$

and we use superscript  $j$  to indicate that the inclusion may be one of many present in the matrix. Clearly, eqns (2.2) and (2.3) now allow us to add up the changes in energy due to expansion of an appropriate number of inclusions, thereby computing how the total elastic strain energies (and hence the moduli) change from  $G^s$ ,  $\nu^s$ , viz. as inclusions "swell" to form the final composite matrix.

Considerable simplification of eqn (2.2) may be achieved by choosing the Cartesian reference system along the principal axes of the ellipsoid (magnitudes  $a_1$ ,  $a_2$ ,  $a_3$  respectively); in the isotropic case most of the components of  $S_{ijkl}$  then become zero and only entries of the form  $S_{JJJJ}$ ,  $S_{JJKK}$ ,  $S_{JKJK} = S_{KJJK} = S_{KKJJ}$  remain.† As Eshelby remarks, this implies that shears of any particular sense (e.g.  $\epsilon_{12}^T$ ) produce shears of the same sense only ( $\epsilon_{12}^A$ ) while extension strains are completely uncoupled from shear-strains. In fact, this same remark applies further to the relation between  $\epsilon_{ij}^T$  and  $\epsilon_{ij}^A$  (eqn 2.2): the consequence is a very simple relation between shear strains‡

$$\epsilon_{JK}^T = (\tilde{G} - G^*) \epsilon_{JK}^A / [2(G^* - \tilde{G}) S_{JKJK} + \tilde{G}], \quad J \neq K. \quad (2.4)$$

However, there is still an awkward coupling between extension strains which must be resolved before eqn (2.3) can be computed. This dependent triplicate in eqn (2.2) can be simply rewritten as a  $3 \times 3$ -matrix equation of the kind

$$[M_{JK}] \{ \epsilon_{KK}^T \} = [R_{JK}] \{ \epsilon_{KK}^A \}. \quad (2.5)$$

The elements of  $[M]$  and  $[R]$  are readily extracted from eqn (2.2), namely

$$M_{JK} = (\lambda^* - \tilde{\lambda}) S_{JJKK} + 2(G^* - \tilde{G}) S_{JKJK} + \tilde{\lambda} + 2\tilde{G} \delta_{JK}, \quad R_{JK} = (\tilde{\lambda} - \lambda^*) + 2(\tilde{G} - G^*) \delta_{JK}. \quad (2.6)$$

The only hitherto undefined quantities appearing are the shape factors

$$S_{JJJJ} = [3a_j^2 I_{JJ} + (1 - 2\nu) I_j] (a_1 a_2 a_3) / 4(1 - \nu) \quad (2.7a)$$

†Capital indices will not follow the Einstein notation of summation when repeated.

‡We note that [4] appears to have errors here (ref. eqn 6) and elsewhere. It also explicitly inverts eqn (2.5) for  $\epsilon_{KK}^T$ ; we retain a simpler structure in order to bring out the significance of intermediate results such as actual stresses induced in inclusions (especially for flat extremes, e.g. as practically relevant for calibration of "flatjacks" in rock engineering).

$$S_{JJKK} = [a_K^2 I_{JK} - (1 - 2\bar{\nu}) I_J](a_1 a_2 a_3) / 4(1 - \bar{\nu}) \tag{2.7b}$$

$$S_{JKJK} = [(a_J^2 + a_K^2) I_{JK} + (1 - 2\bar{\nu})(I_J + I_K)](a_1 a_2 a_3) / 8(1 - \bar{\nu}) \tag{2.7c}$$

where the integrals  $I_J$ ,  $I_{JJ}$  and  $I_{JK}$  of [22, 23] are given, in mnemonically more efficient notation, by the following comprehensive formulae

$$I_{n_j m_K} = \int_0^\infty du / (a_j^2 + u)^n (a_K^2 + u)^m \Delta(u), \quad \Delta(u) \equiv \sqrt{(a_1^2 + u)(a_2^2 + u)(a_3^2 + u)}. \tag{2.7d}$$

Here we get  $I_K$  for  $n = 0$ ,  $m = 1$ , or  $I_{KK}$  for  $m = 2$  and  $I_{JK}$  for  $n = 1 = m$ , and so forth. The integrals in eqn (2.7d) are actually just appropriate combinations of elliptic integrals of the first and second kind [22, eqns (3.9)–(3.16), 26, p. 589], and the shape factors in eqn (2.7) have previously been computed in a number of contexts [e.g. 27, Appendices of 28, 29]. Of particular interest are the cases of an oblate spheroid ( $a_1 = a_2 > a_3$ ) or a prolate spheroid ( $a_1 > a_2 = a_3$ ), when the integrals are expressible in terms of one fundamental combination of elliptic integrals; for the oblate or prolate conditions, respectively, those basic integrals are

$$I_1 = I_2 = [a_1^2 \cos^{-1}(a_3/a_1) - a_3 \sqrt{a_1^2 - a_3^2}] / a_1^2 (a_1^2 - a_3^2)^{3/2} \tag{2.8a}_1$$

$$I_2 = I_3 = [a_1 \sqrt{a_1^2 - a_3^2} - a_3^2 \cosh^{-1}(a_1/a_3)] / a_3^2 (a_1^2 - a_3^2)^{3/2}. \tag{2.8b}_1$$

The other integral is defined in the respective cases by

$$I_3 \text{ or } I_1 = 2[(a_1 a_2 a_3)^{-1} - I_2]. \tag{2.8a,b}_2$$

Now the integrals in eqn (2.7) take the form, again for oblate and prolate spheroids, respectively:

$$I_{13} = I_{23} = (I_3 - I_1) / (a_1^2 - a_3^2), \quad I_{12} = I_{13} = (I_2 - I_1) / (a_1^2 - a_3^2), \tag{2.8c}$$

while a general formula to be kept in mind is

$$3I_{JJ} = 2/a_j^3 a_K a_L - I_{JK} - I_{JL} \quad (J \neq K \neq L). \tag{2.8d}$$

**Spherical inclusions.** The simplest example is the case when the inclusion is spherical,  $a_3 \rightarrow a_2 = a_1$ ; by use of eqns (2.8)<sub>2</sub> or careful expansion of eqns (2.8)<sub>1</sub>, we obtain

$$I_1 = I_2 = I_3 = 2/3a^3, \quad I_{11} = I_{22} = I_{12} = I_{13} = I_{23} = 2/5a^5. \tag{2.9a}$$

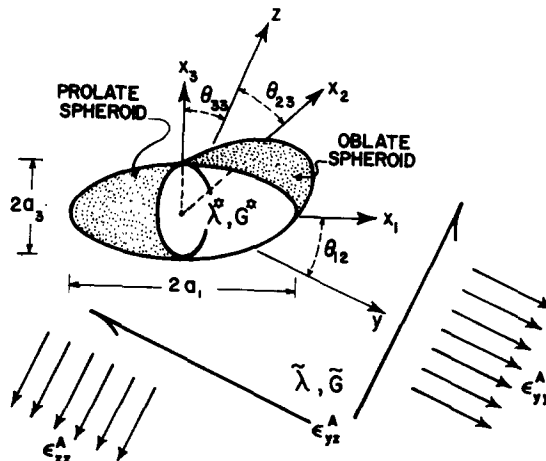


Fig. 1(a). Schematic of a typical spheroidal inclusion (Lamé constants  $\lambda^*$ ,  $G^*$ ) perturbing a strain-field which is remotely uniform in a medium with effective moduli  $\tilde{\lambda}$ ,  $\tilde{G}$ .

Thus the shape-factors in eqns (2.7) become

$$[S_{KKKK}, S_{JJJK}, S_{JKJK}] = [9 + 5(1 - 2\bar{\nu}), 3 - 5(1 - 2\bar{\nu}), 8 - 10\bar{\nu}]/30(1 - \bar{\nu}) \quad (2.9b)$$

from which we can deduce the matrix  $[M]$  in eqn (2.6), having components

$$\begin{Bmatrix} M_{KK} \\ M_{JK} \end{Bmatrix} = \begin{Bmatrix} 2\bar{G} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 28 - 20\bar{\nu} \\ 20\bar{\nu} - 4 \end{Bmatrix} (G^* - \bar{G})/30(1 - \bar{\nu}) + (1 + \bar{\nu})(\lambda^* - \bar{\lambda})/3(1 - \bar{\nu}) + \bar{\lambda}. \quad (2.9c)$$

The case of a remotely applied shear strain  $\epsilon_{JK}^A$  ( $J \neq K$ ) is easiest to solve; indeed, from eqns (2.4) and (2.9b), the result is simply

$$\epsilon_{JK}^T = \beta_1 \epsilon_{JK}^A, \quad \beta_1 \equiv 15(1 - \bar{\nu})/[15(1 - \bar{\nu})g/(g - 1) - 2(4 - 5\bar{\nu})], \quad g \equiv \bar{G}/G^*. \quad (2.9d)$$

The only other independent case is that of remote hydrostatic strain  $\epsilon_{JJ}^A \equiv e^A/3$ ,  $J = 1, 2, 3$ ; inspection of eqn (2.6a), with the form of  $[M]$  in eqn (2.9c), immediately shows that  $\epsilon_{KK}^T$  must be hydrostatic as expected. Thus, only one of the eqns (2.6a) is needed, namely

$$\epsilon_{KK}^T = \alpha_1 e^A, \quad \alpha_1^{-1} \equiv 3\bar{K}/(\bar{K} - K^*) - (1 + \bar{\nu})/(1 - \bar{\nu}), \quad K \equiv \lambda + 2G/3. \quad (2.9e)$$

It is now trivial to compute the energy perturbation due to the presence of a spherical inclusion in a matrix under remote combined shear and hydrostatic stress, namely

$$\Delta E^i = -2[4\bar{G}\beta_1(\epsilon_{JK}^A)^2 + 3\bar{K}\alpha_1(e^A)^2]\pi\Delta[a^{(ij)3}]/3. \quad (2.9f)$$

This can simply be summed over the number of inclusions to get the total energy change, which is therefore clearly proportional to volume concentration when measured per unit representative element of material.

*Flat inclusions.* It is most worthwhile to compute asymptotic expressions for the shape factors as  $a_1/a_3$  grows very large (i.e. for flat or crack-like inclusions): then the integrals are

$$\begin{aligned} I_1 = I_2 &\approx \pi/2a_1^3, \quad I_3 \approx (2a_1 - \pi a_3)/a_1^3 a_3, \quad I_{13} \approx (4a_1 - 3\pi a_3)/2a_1^5 a_3, \\ I_{33} &\approx 2(a_1^2 - 2a_3^2)/3a_1^4 a_3^3, \quad I_{11} \approx 3\pi/8a_1^5 \end{aligned} \quad (2.10a)$$

and eqn (2.7) gives the shape factors

$$\begin{aligned} S_{1111} = S_{2222} &\approx \frac{\pi a_3}{32a_1} \left[ \frac{13 - 8\bar{\nu}}{1 - \bar{\nu}} \right], \quad S_{3333} \approx 1.0 - \frac{\pi a_3}{4a_1} \left( \frac{1 - 2\bar{\nu}}{1 - \bar{\nu}} \right) \\ S_{3322} = S_{3311} &\approx [\bar{\nu} - \pi a_3(1 + 4\bar{\nu})/8a_1]/(1 - \bar{\nu}) \\ S_{1122} = S_{2211} &\approx \frac{\pi a_3}{32a_1} \left[ \frac{8\bar{\nu} - 1}{1 - \bar{\nu}} \right], \quad S_{1133} = S_{2233} \approx -\frac{\pi a_3}{8a_1} \left( \frac{1 - 2\bar{\nu}}{1 - \bar{\nu}} \right) \\ S_{1212} = S_{2121} &\approx \frac{\pi a_3}{32a_1} \left[ \frac{7 - 8\bar{\nu}}{1 - \bar{\nu}} \right], \quad S_{2323} = S_{1313} \approx 0.5 - \frac{\pi a_3}{8a_1} \left( \frac{2 - \bar{\nu}}{1 - \bar{\nu}} \right). \end{aligned} \quad (2.10b)$$

Components of matrix  $[M]$ , eqn (2.5), are now very readily deduced from eqn (2.6).

For our purposes, the most general case of interest is that of an external strain state  $\epsilon_{11}^A = \epsilon_{22}^A = e^A/3$ ; we seek  $\epsilon_{KK}^T$  from eqn (2.5), which now reduces to two coupled conditions:

$$\begin{aligned} 2[\bar{\lambda} + \bar{G} + \bar{\nu}(\lambda^* - \bar{\lambda})/(1 - \bar{\nu})]\epsilon_{11}^T + [\lambda^* - \pi a_3(1 - 2\bar{\nu})(2\lambda^* - 2\bar{\lambda} + G^* - \bar{G})/4a_1(1 - \bar{\nu})]\epsilon_{33}^T \\ = [\bar{K} - K^*]e^A + (\bar{\lambda} - \lambda^*)(\epsilon_{33}^T - e^A/3) \end{aligned} \quad (2.11a)_1$$

$$2[\tilde{\lambda} + \bar{\nu}(\lambda^* - \tilde{\lambda} + 2G^* - 2\tilde{G})/(1 - \bar{\nu})]\epsilon_{11}^T + [\lambda^* + 2G^* - \pi a_3(\lambda^* - \tilde{\lambda} + G^* - \tilde{G})(1 - 2\bar{\nu})/2a_1(1 - \bar{\nu})]\epsilon_{33}^T = (\tilde{K} - K^*)e^A + [\tilde{\lambda} - \lambda^* + 2\tilde{G} - 2G^*](\epsilon_{33}^A - e^A/3). \quad (2.11a)_2$$

Here we have retained terms of order  $a_3/a_1$  only if they are to affect results even when  $G^*$  or  $\lambda^*$  grow very small. The general solution of eqn (2.11a) is straightforward and takes the form:

$$\epsilon_{11}^T = f_1 e^A + f_2 (\epsilon_{33}^A - e^A/3), \quad \epsilon_{33}^T = f_3 e^A + f_4 (\epsilon_{33}^A - e^A/3),$$

where

$$[f_1, f_2] = [\tilde{K} - K^*, \tilde{\lambda} - \lambda_g^*](1 - \bar{\nu})/(1 + \bar{\nu})(\lambda^* + 2G^*)g \quad (2.11b)$$

$$f_3 \equiv [g(1 + \bar{\nu}) - 2\bar{\nu}]f_1/(1 - \bar{\nu}), \quad f_4 \equiv (g - 1) + [g(1 + \bar{\nu}) - 2\bar{\nu}]f_2/(1 - \bar{\nu}).$$

It is especially interesting to compute the resulting stress-state in the inclusion

$$\left\{ \begin{array}{c} \sigma_{11} \\ \sigma_{33} \end{array} \right\}^{\text{inc}} = \left[ \lambda^* \frac{3\tilde{K} + 4G^*}{\lambda^* + 2G^*} + 2G^* \right] e^A/3 + (\epsilon_{33}^A - e^A/3) \left[ \lambda^* \frac{\tilde{\lambda} + 2\tilde{G}}{\lambda^* + 2G^*} + \left\{ \begin{array}{c} 0 \\ 2G^* \end{array} \right\} \right] \quad (2.11c)$$

which becomes hydrostatic and equal to the normal applied stress  $\sigma_{33}^A$  whenever  $G^*/\tilde{G}$  is negligible (a verification of the assumption commonly made in this context).

On the other hand, we are also interested in a remote shear strain  $\epsilon_{JK}^A$  ( $J \neq K$ ) and then we find, using eqns (2.4) and (2.10b), that

$$\epsilon_{JK}^T \approx \epsilon_{JK}^A [g/(g - 1) - 2S_{JKJK}], \quad J \neq K. \quad (2.11d)$$

The energy associated with the presence of the inclusion under any combination of the distinguishable remote stresses  $\sigma_{12}^A$ ,  $\sigma_{13}^A$ ,  $\sigma_{kk}^A$  and  $\sigma_{33}^A$  may now be estimated directly from eqn (2.3), with the aid of solutions given in eqns (2.11b–d).

*Soft flat inclusions.* It will be noticed immediately that an expected degeneracy develops in eqns (2.11) when the inclusion stiffnesses ( $\lambda^*$ ,  $G^*$ ) become smaller than those of the matrix ( $\tilde{\lambda}$ ,  $\tilde{G}$ ) by factors comparable to  $a_1/a_3$ ; it is then no longer adequate to neglect the second-order terms in eqn (2.11a). For instance, let us choose the extreme case  $G^* = \pi\gamma a_3 \tilde{G}/4a_1$  (but  $\lambda^*$  still finite, viz.  $\nu^* = 0.5$ ); then, most obviously,  $\epsilon_{JK}^T$  ( $JK = 13, 23$ ) goes unbounded in eqn (2.11d) unless we add in the additional second-order term in eqns (2.10b), with the result

$$\epsilon_{JK}^T \approx 4a_1 \epsilon_{JK}^A / [\gamma + (2 - \bar{\nu})/(1 - \bar{\nu})] \pi a_3. \quad (2.12a)$$

However, all of the other strain ratios are of order unity: eqn (2.11d) gives the third shear strain ( $\epsilon_{12}^T \approx \epsilon_{12}^A$ ) and eqn (2.11b) simplifies to

$$\epsilon_{11}^T \approx -(1 - \bar{\nu})(\epsilon_{33}^A - e^A/3)/(1 + \bar{\nu}), \quad (2.12b)$$

$$\epsilon_{33}^T \approx [(\tilde{K} - K^*)e^A + (3\tilde{K} - \lambda^*)(1 - \bar{\nu})(\epsilon_{33}^A - e^A/3)/(1 + \bar{\nu})]/\lambda^*.$$

Nevertheless, we must emphasize that eqn (2.12b) arises after an elimination in eqn (2.11a) which generates the intermediate equation

$$\pi a_3 [\gamma + 2(1 + \bar{\nu})/(1 - \bar{\nu})] \epsilon_{33}^T / 8a_1 \approx [(\epsilon_{33}^A - e^A/3) + \epsilon_{11}^T (1 + \bar{\nu})/(1 - \bar{\nu})], \quad (2.12c)$$

and eqn (2.12b) must be modified if ever  $\epsilon_{33}^T$  becomes of order  $a_1/a_3$ . Equations (2.11b) and (2.12b) are again a formal indication of another statement commonly made from simple physical considerations (e.g. in [6]) that the surfaces of a flat inclusion are “glued in place” (viz. exhibit negligible convergence) due to the presence of material (e.g. fluid) with non-negligible bulk modulus; such sites will wreak little alteration in  $\tilde{K}$ .

Clearly, the expression for  $\epsilon_{33}^T$  in eqn (2.12b) cannot be valid if  $\lambda^*$  becomes less than  $\tilde{\lambda}$ ,  $\tilde{G}$  by factors of order  $a_1/a_3$ ; under that condition,  $a_3 \epsilon_{33}^T / a_1$  is not negligible in eqn (2.12c) and we must

rather insert back into eqn (2.11a) to get, employing  $\lambda^* \equiv \pi \Lambda \tilde{\lambda} a_3 / 4a_1$ ,

$$\epsilon_{33}^T \approx 4g_1[(\epsilon_{33}^A - e^A/3) + (1 + \bar{\nu})e^A/3(1 - \bar{\nu})]a_1/\pi a_3, \quad \epsilon_{11}^T = \epsilon_{22}^T \approx g_2(\epsilon_{33}^A - e^A/3) + g_3e^A/3, \quad (2.13a)$$

where

$$g_1 \equiv (\tilde{\lambda} + 2\tilde{G})(1 - \bar{\nu})/[2\tilde{G}(1 + \gamma - \gamma\bar{\nu}) + \Lambda\tilde{\lambda}(1 - \bar{\nu})] \\ g_2 \equiv [1 - 2\bar{\nu} + 2\gamma(1 - \bar{\nu})]/2(1 + \bar{\nu}), \quad g_3 \equiv g_1[\gamma + (1 - 2\bar{\nu})/2(1 - \bar{\nu})]. \quad (2.13b)$$

These results establish a very important conclusion, namely that the strains in the remote stretch direction (parallel to the minor axis of the flat oblate low-modulus spheroid) are indeed intensified in inverse proportion to the flatness ratio  $a_3/a_1$ ; thus, for both tension and the analogous shear result in eqn (2.12a), the energy in eqn (2.3) does not grow vanishingly small with  $a_3/a_1$  but rather reaches a finite limit depending on matrix and inclusion moduli. A case of particular convenience for later computations is that of uniaxial remote tension  $\sigma_{33}^A$  (i.e.  $e^A \approx -3\bar{\nu}\epsilon_{33}^A$ ), for which the transformation strains are  $\epsilon_{11}^T (= \epsilon_{22}^T)$  of order unity and

$$\epsilon_{33}^T \approx 4a_1(1 - \bar{\nu})\sigma_{33}^A/\pi a_3[2\tilde{G}(1 + \gamma - \gamma\bar{\nu}) + \Lambda\tilde{\lambda}(1 - \bar{\nu})]. \quad (2.13c)$$

It is readily shown that purely in-plane stresses ( $\sigma_{11}^A, \sigma_{22}^A$ ) induce strains  $\epsilon_{JK}^T$  of order  $\epsilon_{JK}^A$  so that only  $\sigma_{J3}^A$  ( $J = 1, 2, 3$ ) actually involves non-negligible energy changes as  $a_3/a_1 \rightarrow 0$ .

(ii) *The problem of orientation distributions.* Since the generic inclusion just considered can be arbitrarily oriented with respect to the remotely applied stress field, we must integrate and normalise over all possible orientations to get an appropriately averaged energy perturbation.† Here we have the simplest case of an isotropic (viz. directionally unbiased or random) distribution, although the methodology naturally carries over quite readily to special kinds of anisotropy (e.g. more realistic orthotropy), particularly when these are induced by the nature of inclusion orientations. For the isotropic composite the problem is trivial in the case of hydrostatic stress but a remote tensile‡ stress (say  $\sigma_{xx}$ ) induces stress components, in the natural coordinate system of the spheroidal inclusion, which we list here in terms of their energy-related distinguishable quadratics

$$\sigma_{12}^2 = C_{11}^2 C_{21}^2 \sigma_{xx}^2, \quad \sigma_{33}^2 = C_{31}^4 \sigma_{xx}^2, \quad (\sigma_{11} + \sigma_{22})^2 = (C_{11}^2 + C_{21}^2)^2 \sigma_{xx}^2 \\ \sigma_{13}^2 + \sigma_{23}^2 = C_{31}^2 (C_{11}^2 + C_{21}^2) \sigma_{xx}^2 = (\sigma_{11} + \sigma_{22})\sigma_{33} \quad (2.15a)$$

where  $C_{ij}$  ( $\equiv \cos \theta_{ij}$ , Fig. 1a) is the usual "direction-cosine" or projection of the  $i$ th natural coordinate of the inclusion of the  $j$ th coordinate of the fixed  $x, y, z$  system in which the remote stress is expressed. Very simply, then, we need the unbiased average over all their values, of the various coefficients appearing in eqn (2.15a). The computations are quite simple and lead to

$$\langle C_{11}^2 C_{21}^2 \rangle = \langle C_{31}^2 C_{11}^2 \rangle = \langle C_{31}^2 C_{21}^2 \rangle = 1/15, \quad \langle C_{31}^4 \rangle = \langle C_{11}^4 \rangle = \langle C_{21}^4 \rangle = 1/5 \quad (2.15b)$$

where the carats in the present case imply an integrated average of coefficients with equal probability of finding any of the possible values of each of the angles with cosines appearing in  $C_{ij}$ . For ordered site distributions leading to anisotropic composite moduli, the quantities in eqn (2.15b) would have to be weighted with the greater probabilities of finding  $x_1, x_2, x_3$  axes (natural to the inclusion) in particular orientations with respect to any fixed  $x, y, z$  axis.

The general result obtained, when we considered the energy perturbation due to the presence of a spheroidal inclusion in an arbitrary remote stress field, was (in terms of constants  $\gamma_K$  deducible from Section (i)) that

$$\Delta E^i = -\frac{1}{2}[\gamma_1(\sigma_{11}^A + \sigma_{22}^A)^2 + \gamma_2\sigma_{12}^A + \gamma_3(\sigma_{13}^A + \sigma_{23}^A) + \gamma_4\sigma_{33}^A + \gamma_5\sigma_{33}^A(\sigma_{11}^A + \sigma_{22}^A)]\Delta V^i. \quad (2.16a)$$

†This orientation averaging seems to be neglected by some authors at the stage of explicit modulus computations.

‡A slightly less transparent procedure may be followed for remote shear stress (e.g.  $\sigma_{xy}^A$ ).



When we are considering a spherical inclusion or a hydrostatic remote stress this may be applied directly by replacing  $\sigma_{11}$  by  $\sigma_{xx}$ ,  $\sigma_{12}$  by  $\sigma_{xy}$ , etc. but the more general case requires us to use eqns (2.15a) and (2.15b) for uniaxial stress and the result is

$$\Delta E^j = -\frac{1}{2}(8\gamma_1 + \gamma_2 + 2\gamma_3 + 3\gamma_4 + 2\gamma_5)\sigma_{xx}^2 \Delta V^j / 15. \tag{2.16b}$$

REPRESENTATIVE MICROSTRUCTURAL ELEMENT, VOLUME V

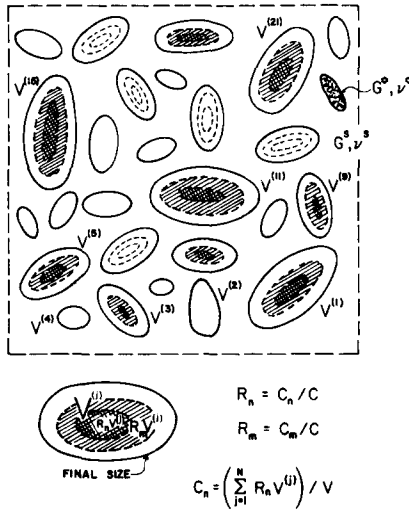


Fig. 1(b). Representative choice for a generic sample of microstructure, considered adequate to constitute an isotropic continuum element; the process is depicted whereby inclusion concentration  $C$  is increased by self-similar stages as the volume  $V^{(j)}$  of each site is expanded in proportion to the rise in  $C$ .

(iii) *Choice of exterior moduli and incremental process of inclusion addition.* We propose here a philosophy for computing the overall modification of solid matrix moduli, due to the addition of any suitable density of second-phase inclusions, which differs significantly from that typically adopted in previous literature on this subject.† Let us suppose the remotely applied stress is  $\sigma_{XY}^A$  (so denoted to express a component in a fixed system,  $X, Y = x, y$  or  $z$ ) and that eqns (2.16a, b), with the aid of Section (i), have led to the energy perturbations (using  $M$  to denote the compliance conjugate to  $\sigma_{XY}$ )

$$\Delta E^j = -\frac{1}{2} \tilde{M} \Gamma \sigma_{XY}^A \Delta V^j, \quad \Gamma = \Gamma(\tilde{\lambda}/\tilde{G}, \lambda^*/G^*, \tilde{\lambda}/\lambda^*, a_2/a_1, a_3/a_1). \tag{2.17a}$$

Then the change in energy per representative volume element of composite ( $V$ ), in which there are a sufficient number  $N$  of arbitrarily-oriented inclusions (Fig. 1b), is

$$\frac{1}{2} (\Delta \tilde{M}) \sigma_{XY}^A = -\frac{1}{2} \tilde{M} \Gamma \sigma_{XY}^A \Delta C, \quad \Delta C \equiv \sum_{j=1}^N \Delta V^j / V. \tag{2.17b}$$

We easily recognize that a change in the ambient moduli accompanies each increment in the volume concentration‡ of inclusions  $C$  and that the limit of small increments yields

$$\int_{M'}^M d\tilde{M} / \tilde{M} = \ln(M/M') = - \int_{C'}^C \Gamma dC. \tag{2.17c}$$

†The exceptions in [30, 34] have come to my attention; the present approach seems to be distinguished by potential for descriptive generality and confirmation by more rigorous formulations [e.g. 33].

‡Notice, however, that the limit of very flat (viz. negligible volume) inclusions requires that  $V^j$  (and thence  $C$ ) be redefined as volume  $\times$  major axis/minor axis, as shown by results in Section (i).

Equation (2.17c) differs from the conventional expressions for effects of inclusions on moduli [e.g. 1, 4, 6, 7] most obviously in treating inclusion addition as an incremental process in which, for computation purposes, the self-consistent approximation is applied successively at each of a continuous series of inclusion densities between the initial density  $C^s$  (with known compliance  $M^s$ ) and the final concentration  $C$  (with unknown compliance  $M$ ). It is, thus, as if a desired orientation distribution is preserved throughout the process but the volume of the inclusions are "pumped up" synchronously from initial to final sizes. The results of this artifact, which removes the ambiguity in choice of effective moduli for the equivalent continuum surrounding any generic inclusion, will depart substantially from those where eqn (2.17b) is applied in only a single increment ( $\Delta\tilde{M}$ ) from initial to final moduli (with either of these,  $M^s$  or  $M$ , being used as the effective compliance). Thus, the assumption of previous authors that the dilute concentration idealization (choice of  $M^s$  for exterior compliance) is best extended to the more general case of "inclusion interaction"† by employing  $M$  for the exterior compliance, in a single-step transition from  $C^s$  to  $C$ , is seen to be essentially unfounded; they may, occasionally, produce estimates which lie closer to reality [e.g. 31] but this would have to be regarded as a serendipitous coincidence with the results of discrete interaction effects at high densities.

*Example 1. Shear modulus of composite with rigid/porous spheres.* From eqns (2.9f, d), we observe that the energy function in eqn (2.17a) is

$$\Gamma_G \equiv -15(1 - \bar{\nu})/[15(1 - \bar{\nu})g/(g - 1) - 2(4 - 5\bar{\nu})], \quad M \equiv G^{-1}, \quad g \equiv \tilde{G}/G^*. \quad (2.18a)$$

For two simple limits,  $g \rightarrow 0, \infty$ ,  $\Gamma$  becomes independent of  $M$ ; however,  $\Gamma$  is not independent of  $C$  unless  $\bar{\nu}$  is unaffected by inclusion addition. For analytical demonstration purposes, suppose the matrix and inclusion material is relatively incompressible‡ ( $\bar{\nu} \approx 0.5$ ): then  $\Gamma$  becomes  $5/4$  and  $-5/3$ , respectively, for the limits just mentioned. More generally, if  $\bar{\nu}$  does happen to remain relatively constant, the consequence of eqn (2.17c) is that shear modulus  $G^s$  changes, with addition of spherical inclusion concentration  $C - C^s$ , to a new value given by

$$\ln(\tilde{G}/G^s) = 15(1 - \bar{\nu})(C - C^s) / \left\{ \begin{array}{l} 8 - 10\bar{\nu} \\ 5\bar{\nu} - 7 \end{array} \right\}, \quad g = \left\{ \begin{array}{l} 0 \\ \infty \end{array} \right\}. \quad (2.18b)$$

This equation implies an exponential increase or decrease of shear modulus with inclusion concentration; clearly, the fact that moduli are not zero or infinity,§ respectively, for unit change in volume concentration is an aspect which reveals limitation on the validity of the assumptions for high concentrations viz. intersection of inclusions. However, we emphasize that the theory does not recognize such intersection (viz. fracture or rigidity) and there is no basis in the theory for the assertions (e.g. [6]) that zero moduli are to be expected at sufficiently high densities of porosity.

*Example 2. Tensile modulus of composite with flat shear-compliant inclusions.* By referring to eqn (2.12a), we conclude that the appropriate translation of eqn (2.3) into eqn (2.17a) involves the use of energy perturbations due to shear only (since  $\lambda^*$ , comparable to  $\bar{\lambda}$ , seals the inclusion faces), resulting in

$$\Delta E^i = \{4a_1(\sigma_{13}^A + \sigma_{23}^A)/2\tilde{G}\pi a_3[\gamma + (2 - \bar{\nu})/(1 - \bar{\nu})]\}\Delta(4\pi a_1^2 a_3/3)^{(j)}. \quad (2.19a)$$

It is now easy to identify the consequent relation, by eqn (2.16b), to an imposed tensile stress  $\sigma_{xx}^A$ , namely

$$\Delta E^i = \{8(1 + \bar{\nu})/15\pi[\gamma + (2 - \bar{\nu})/(1 - \bar{\nu})]\tilde{E}\}\sigma_{xx}^A \Delta(4\pi a_1^3/3)^{(j)}. \quad (2.19b)$$

†Clearly a misnomer for, more precisely, the condition that inclusion density is great enough to alter composite moduli by fractions of order unity: strong discrete interaction actually renders the surrounding continuum assumption untenable, unless [21] clusters of inclusions are employed in computing single-site energy perturbations.

‡The only other physically realistic choice is  $\bar{\nu} \approx 0.2$ ; this gives equal positive and negative coefficients (+2, -2) in eqn (2.18b) and similar behavior for  $M = K^{-1}$ , so  $\bar{\nu}$  does remain constant with inclusion addition.

§Notice that  $G$  must become comparable to  $\lambda$  eventually, which formally invalidates the derivation for  $\bar{\nu} \approx 0.5$ .

Here we are clearly adopting an inclusion growth at constant "aspect" ratio  $a_3/a_1$  and have employed the equivalent volume  $4\pi a_1^3/3$ , rather than the actual volume (much smaller and unrepresentative of inclusion density). By comparison with eqn (2.17a), we can now deduce

$$\Gamma_E \approx -16(1 + \bar{\nu})/15\pi[\gamma + (2 - \bar{\nu})/(1 - \bar{\nu})], \quad \ln(E/E_s) \approx \mu_E(C - C_s) \quad (2.19c)$$

where the latter rough estimate is rigorously valid only for constant  $\bar{\nu}$ .

However, rather than  $\bar{\nu}$  constant, the more reasonable statement<sup>†</sup> is that flat inclusions of  $\lambda^*$  comparable to  $\bar{\lambda}$  do not appreciably alter the bulk modulus  $K^*$ ; it follows that  $\bar{E}/3(1 - 2\bar{\nu}) = \bar{K}$  is essentially constant and a more accurate estimate than eqn (2.19c) can then be worked out, having the form

$$\ln[\bar{E}^2(3 - \bar{E}/3K^*)(1 + \bar{E}/3K^*)]_{E_s}^E = -16(C - C^s)/15\pi. \quad (2.19d)$$

Although this may appear to give totally different predictions from eqn (2.19c)<sub>2</sub> it will later (in Fig. 2) be clear that the estimates are, in fact, quite close and this feature will be found to apply over a range of conditions: thus, eqns (2.18b) and (2.19c)<sub>2</sub> do give a very good impression of the modulus variation with inclusion concentration, one which is usually not drastically altered by (fortuitous) analytical results like eqn (2.19d) (and its analogue for spheres, using constant  $\bar{K}$  in eqn 2.18a), or by the more detailed numerical results to be discussed next.

*Example 3. Shear and bulk moduli with spherical inclusions.* If we are to find the concentration-dependence of  $\bar{\nu}$  for use in eqn (2.18a), then we must simultaneously solve for the changes in  $\bar{K}$  (with varying inclusion density) by identifying  $\Gamma_K$ , corresponding to  $\sigma_{XY} = \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \bar{K}e^A$  in eqn (2.17a); from eqns (2.9e, f) it clearly takes the form

$$\Gamma_K = -3/[3k/(k - 1) - (1 + \bar{\nu})/(1 - \bar{\nu})], \quad k \equiv \bar{K}/K^*, \quad M \equiv K^{-1}. \quad (2.20a)$$

For the special cases  $k = 0, \infty$  and a (generally artificial) fixed choice of  $\bar{\nu}$ , we again get an exponential growth of moduli, analogous to eqn (2.18b), namely

$$\ln(K/K^s) = 3(1 - \bar{\nu})(C - C^s) \begin{cases} 1 + \bar{\nu} \\ 4\bar{\nu} - 2 \end{cases}, \quad k = \begin{cases} 0 \\ \infty \end{cases}. \quad (2.20b)$$

However, the realistic physical situation does not usually maintain fixed  $\bar{\nu}$  (a better estimate being  $\bar{G}$  fixed, for some instances); thus, in general, we must rather use  $\Gamma_G$  in eqn (2.18a) and  $\Gamma_K$  in eqn (2.20a) to solve the simultaneous differential equations

$$d\bar{G}/\bar{G} = \Gamma_G dC; \quad d\bar{K}/\bar{K} = \Gamma_K dC; \quad \bar{K} \equiv 2\bar{G}(1 + \bar{\nu})/3(1 - 2\bar{\nu}) \quad (2.20c)$$

on the interval over which  $C$  varies from  $C^s$  to its final value. These computations have been conducted for a variety of  $\nu^s$ ,  $g^s$ ,  $k^s$  and the resulting variation of  $G$ ,  $K$  (or  $\nu$  marked occasionally on plots) is shown in Fig. 2.

*Example 4. Tensile and bulk moduli with flat inclusions.* A coupling of the incremental moduli equations, analogous to that in eqns (2.20c), must also pertain to the opposite end of the ellipticity spectrum, namely the case where  $a_3/a_1$  is very small while  $G^*$  and/or  $\lambda^*$  is smaller than  $\bar{G}$  (or  $\bar{\lambda}$ ) by factors of order  $a_1/a_3$ . We already have considered the special case where  $\lambda^*$  is comparable to  $\lambda$  (eqns 2.19d) so we now treat the more general situation where both  $G^*$  and  $\lambda^*$  are small. Extraction of  $\gamma_3, \gamma_4$  from eqns (2.12a, 13a) and insertion in eqn (2.16b) produces the incremental relations for tensile stress, namely

$$d\bar{E}/\bar{E} dC \equiv \Gamma_E = -4(1 + \bar{\nu})\{4/[\gamma + (2 - \bar{\nu})/(1 - \bar{\nu})] + 3(1 - \bar{\nu})/[1 + (1 - \bar{\nu})(\gamma + \Lambda\bar{\nu}/(1 - 2\bar{\nu}))]\}/15\pi. \quad (2.21a)$$

<sup>†</sup>It is worth noting that  $\bar{\nu} = 0.5, 0.0$  are, in fact, values at which Poisson ratio can remain constant, just as  $\bar{\nu} = 0.5, 0.2$  were specially admissible for spherical inclusion addition.

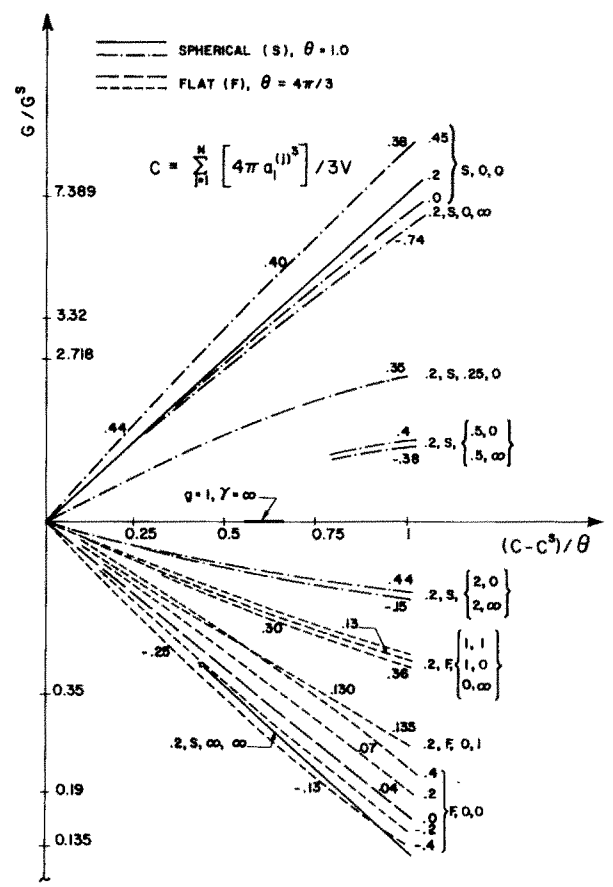


Fig. 2(a). Dependence of shear modulus on inclusion concentration: parameters are initial Poisson ratio, equi-axedness, inclusion shear and bulk moduli ( $\nu^s$ ,  $S(F)$ ,  $g^s(\gamma^s)$ ,  $k^s(\Lambda^s)$ , respectively). For spheres (S) the curves for  $g > 1$  are similar to those shown for  $g < 1$ . Isolated numbers indicate variation in  $\nu$ . Curves chosen bound interesting range or indicate trends, especially for flat inclusions (F).

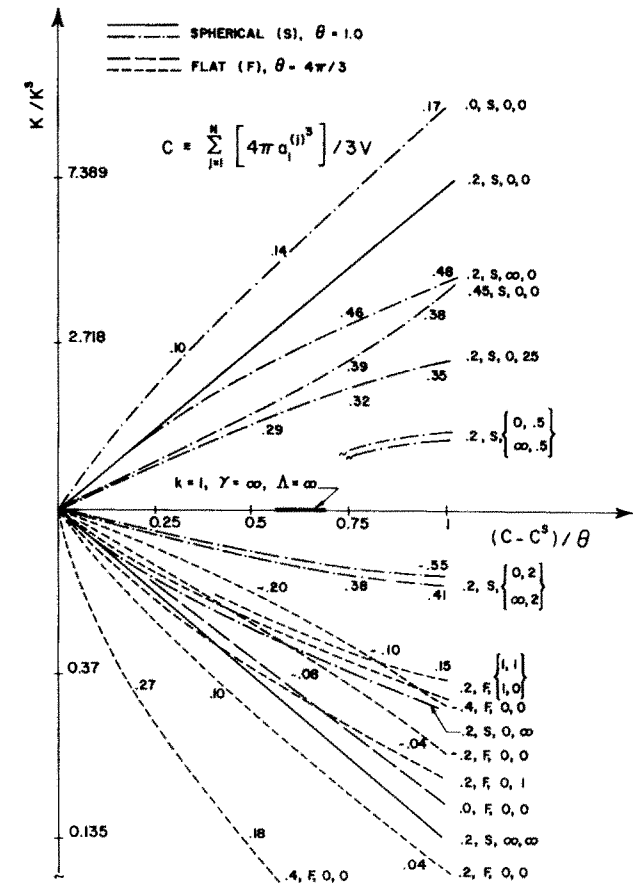


Fig. 2(b). Bulk modulus variation with concentration  $C$  and parameters  $\nu^s$ ,  $S$  or  $F$ ,  $g^s$  or  $\gamma^s$ ,  $k^s$  or  $\Lambda^s$  (reported in that order). Behavior for  $k > 1$  is almost mirror image of that shown for  $k < 1$  and chosen curves indicate bounds or trends: more complete results have been worked out and a simple program (on programmable pocket calculator) is available to compute the curves for any specified initial values of the solid and inclusion material parameters.

For the case of hydrostatic stress (corresponding to strain  $e^A/3$  in each direction) the only component contributing appreciably to the energy is that of uniaxial stressing perpendicular to the "plane of the inclusion"; thus, from either of eqns (2.13a, c), we can obtain the required function  $\Gamma_K$  for  $M = K^{-1}$  in eqn (2.17c). The resulting set of simultaneous differential equations, analogous to eqn (2.20c) and with  $\Gamma_E$  defined by eqn (2.21a), take the now familiar form

$$\begin{aligned} d\tilde{E}/\tilde{E} &= \Gamma_E dC, & d\tilde{K}/\tilde{K} &= \Gamma_K dC; \\ \Gamma_K &\equiv 4(1-\bar{\nu})(1+\bar{\nu})/3\pi(1-2\bar{\nu})\{1+(1-\bar{\nu})[\gamma+\Lambda\bar{\nu}(1-2\bar{\nu})]\}. \end{aligned} \quad (2.21b)$$

Again, we have computed the implications of eqn (2.21b) for a variety of  $\nu^s$ ,  $\gamma^s$ ,  $\Lambda^s$ ; the resulting dependence of  $G, K$  (and occasionally  $\nu$ ) on  $C - C^s$  is shown in Fig. 2.

(iv) *Extension of spheroidal results and related computations.* Inspection of eqns (2.12a) and (2.13c) and use of eqn (2.1) shows that the relative displacement  $\delta$  between flat faces of the inclusion, namely  $2a_3\epsilon_{JK}^{\text{inc}}$  for the constant inclusion strain (hence approx  $a_3\epsilon_{13}^T$  and  $2a_3\epsilon_{33}^T$ , respectively), takes the form

$$\delta = \frac{2a_1(1-\bar{\nu})}{\pi} \left\{ \frac{\sigma_{13}^A/\tilde{G}[2-\bar{\nu}+\gamma(1-\bar{\nu})]}{2\sigma_{33}^A/[2\tilde{G}(1+\gamma-\gamma\bar{\nu})+\Lambda\tilde{\lambda}(1-\bar{\nu})]} \right\} \quad (2.22)$$

for an externally applied shear stress  $\sigma_{13}^A$  (or  $\sigma_{23}^A$ ) or tensile stress  $\sigma_{33}^A$ . When  $\gamma \approx 0 \approx \Lambda$  we recognize the results for a "penny-shaped" crack, familiar in the fracture mechanics literature (e.g. [32]): this immediately suggests an extension to more general crack shapes by simply using available solutions for  $\delta$  (e.g. those in [32] for plane elliptical internal crack surfaces), thus avoiding the computation of the general elliptic integrals in eqn (2.7d), which are as tractable as eqns (2.8) only for that special case.

However, Budiansky and O'Connell[6, 7] have explored this general area of specialization quite thoroughly: their estimate of the energy perturbation due to the presence of distributed elliptical cracks, eqns (8), (25) and (33), is readily substituted into our differential eqns (2.17). In fact, the only distinction between our equations would be a redefinition of the kind

$$\{\bar{L} - L = -L\Gamma_L(\bar{\nu}, a_1/a_2)(C - C^s)\} \rightarrow \{d\bar{L} = -\bar{L}\Gamma_L(\bar{\nu}, a_1/a_2) dC\}, \quad (2.23)$$

where  $L$  stands for any modulus to be determined; for instance, eqns (35, 39) in [7] give  $\Gamma_K$  and  $\Gamma_E$  respectively. Thus, their methodology [l.h.s. of eqn (2.23)] is deceptively like one term in a Taylor expansion of the exponential variation of moduli which results from a rigorous alteration of crack density by differential steps.† Nevertheless, Refs. [6, 7] do demonstrate the important (energetically predictable) result that shape of crack surface does not substantially affect the variation of moduli with crack density (e.g. see their Fig. 7): thus, our ability to include  $\gamma$  and  $\Lambda$  for elliptic cracks [after computing the general integrals in eqn (2.7d)] does not have sufficient physical importance to warrant its pursuit here. Likewise, it is perfectly straightforward to deduce and solve the differential equations (corresponding to 2.17) for the other extreme of prolate spheroids viz. long cylindrical inclusions or pores (of the kind often assumed in somewhat artificial porous media models e.g. [14, 15]); although some such computations, requiring the modifications implied in eqn (2.23), have actually been pursued, e.g. [4], we consider that they have little demonstrative value here (e.g. see [24] for some pore results).

#### (b) *Other models: high inclusion densities, asperity contacts*

It is quite clear from the preceding section that our particularly tractable implementation of the self-consistent concept does not extend reliably to very high concentrations (especially of

†We note that this inconsistency in the approach of Ref. [6] has been pointed out also in Ref. [34]. However, all of these authors proceed by integrating the energy release rate around the perimeter of the crack surfaces and they obtain the latter from the squares of stress-intensity factors (through the familiar formula of fracture mechanics), using the effective moduli of the medium to make the conversion: in fact, the crack will be growing locally through a medium with moduli  $G^s, \nu^s$ , and it is rather the usual expression for stress-intensity factor which has to be modified, due to the inclusion character of a crack in a region with moduli  $G^s, \nu^s$  surrounded by an exterior with moduli  $\tilde{G}, \bar{\nu}$  (e.g. see analogous shielding effect in Chap. 5 of [13]). Our present use of Eshelby expressions for energy, apart from allowing ready inclusion of  $\gamma$  and  $\Lambda$ , avoids this bothersome rationalisation (completely ignored in [6, 7, 34]).

very stiff or very flexible inclusions): one obvious (but not necessarily primary) reason is that any generic inclusion eventually senses the discrete presence of nearest neighbors and the effective continuum model of surrounding-microstructure begins to break down. An improvement might be made by establishing the solution for a group of inclusions in an otherwise homogeneous medium (thereby accounting for interaction to some extent). Only a few such solutions are available to us [21] and we consider that a detailed investigation of their consequences is not merited, simply because a more serious problem is the inevitable "clustering" of sites which may create larger effective inclusions (perhaps traversing the sample): except perhaps for comparable solid phases, this linking-up (which is the only possible *rationale* for claims [6, 7] that stiffnesses disappear at high pore densities) obviously falls beyond the scope of all deterministic self-consistent models (especially those employing compact Eshelby-type results). To broaden the perspective just a little, we allude here to a few other major techniques which must not be ignored in attempts to derive adequate estimates for response of porous media.

(i) *Integral equation and discretization techniques.* One option is to attempt explicit solution for the composite response of an actual body containing a distribution of inclusions, pores or randomly oriented crystals (e.g. [8, 35] and [36] for analogous permeability estimates): if the distribution adequately represents the microstructure (in the sense of layout and adequate array of elements) then the effective moduli obtained should closely approximate the real behavior of a representative microstructural element. One efficient basis for such numerical estimates is the establishment of a suitable fundamental solution for misfit strains in the aggregate [e.g. 8]; such misfit strains are, essentially, distributed to simulate the aggregate departures from homogeneous response but no assumptions of uniform inclusion strain are required and the surrounding composite is not *a priori* necessarily regarded as an effective continuum (although estimates obtained thus far have involved exactly such a specialization). Despite the appealing analytical basis for such integral equations, their actual numerical implementation may not have substantial advantages over a simple finite-element computation, in which a sufficient number of suitably oriented elements (with native stiffnesses) is distributed to represent an element with properly parameterised remote stress or strain conditions. High densities of heterogeneity sites will certainly overwhelm the usual appeal of singular integral formulations in (semi-) infinite regions but their generality is still appealing as a starting point for specialization [21]; indeed, accuracy of finite elements will require some ingenious mesh design.

(ii) *Discrete structural connectivity, granular media.* An extreme example, where compact continuum models lose their validity, involves a fibrillous structure (e.g. textiles, laminate clays) for which only a discrete structural analysis (e.g. of a space framework) may provide reliable estimates [21]. Another extreme is the case of a granular solid matrix in which porosity is present due to the inability of grain shapes to completely fill the effective occupied space of the material. Here the assumption of uniform inclusion strain is clearly unrepresentative, since the dominant compliance of the structure comes from the asperity-like contact at the interface between grains. Various *Hertz*-type contact models have been constructed to establish the overall (macroscopic) moduli resulting from a regular array of such singular load-transmitting points in a representative element: some particular computational results are provided in [37, 38, 16] and reviewed in [17]. Dominant features are the strong non-linearity† (with the usual *Hertzian* form), the fact that resulting moduli are not simple combinations of component material moduli, and the oft-neglected potentially intense variation in composite stiffness with fluid moduli (when the latter are comparable to matrix stiffnesses); such observations remind us, especially of the diverse considerations needed to achieve a reliable estimate for the incremental moduli of a heterogeneous aggregate with multiple sources of additional compliance or stiffness (apart from the basic moduli of the constituents). Results in Fig. 2 are, therefore, to be treated as demonstrative only in studying the magnitudes of transitions between the various regimes discussed next.

### 3. REGIMES AND CHARACTERISTIC TIMES OF TRANSITION BETWEEN LIMITING STIFFNESSES

In Section 2 we outlined the techniques for computing the various elastic limits of response as internal local relaxation or global diffusion occurs between sites containing fluidic elements.

†Nonlinearity results also from crack closure in Section 2(a), an effect under separate analysis (toward medium-amplitude wave propagation and stress-induced permeability changes).

In order to appreciate the timescales, on which such distinctions between moduli can be important, we now elaborate on previous discussions[39], identifying some of the typical mechanisms for internal redistribution of stress and estimating their primary (or characteristic) times of occurrence.

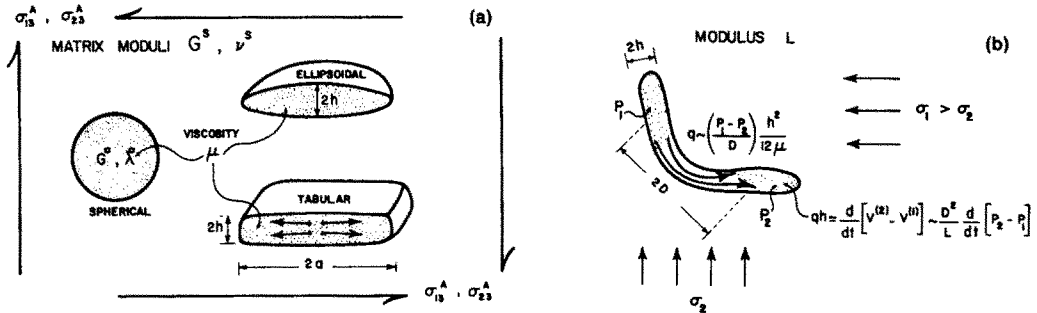


Fig. 3(a). Schematic of the various inclusion shapes of importance in considering the regimes of relaxation for an element under exterior stress; intra-site fluid transport is indicated in the tabular pore, as against uniform shear in ellipsoid. (b). Illustration of computation for microstructural mechanism of time-dependence (which will produce a characteristic time of order  $D^2/h^3$ ): a matter-filled inclusion under major stress  $\sigma_1$  (pressure  $p_1$ ) "squirts" fluid into an adjacent site under lower stress  $\sigma_2$  (and pressure  $p_2$ ).

(i) *Relaxation of completely isolated sites* ( $\hat{G}, \hat{\nu} \rightarrow \bar{G}, \bar{\nu}$ ). Fig. 3(a)

The concept here is that each of an arbitrary number of isolated sites, containing a specific source of time-dependence, independently transfers some (or all) of the stress upon it to the surrounding matrix, thereby altering the amount of elastic stored energy in the stressed continuum element, assumed well-defined as containing a representative distribution of such sites. A specific example is an array of globules of viscous fluid (or solid) in an otherwise elastic matrix: instantaneously these exhibit a stiff elastic response but then they relax with passage of time, for instance even completely shedding the shear-stress upon them. If  $\mu^i$  are representative measures of viscosity for these inclusions and  $L^k$  are the moduli dominating the amount of elastic deformation which the globules will undergo, then these primary characteristic times must be

$$\tau^{(ik)} = (\mu^i/L^k) f_a(a_1^{(j)}/a_3^{(j)}) g_a(s). \tag{3.1a}$$

Here  $a_1^{(j)}$  and  $a_3^{(j)}$  are meant to represent the dominant dimensions of the  $j$ th class of inclusions; they appear because we observe that the amount of shear-straining (rate of which is governed by  $\mu^i$ ), for a given amount of stress-transfer by the inclusion (which  $L^k$  convert to relative offsets across the inclusion), is strongly influenced by the degree to which inclusions are flat or spherical. Further, relaxation times will also be influenced by the overall geometric array of inclusions, denoted mnemonically here as a structure  $s$ , an effect which we will crudely blend into the response moduli  $L^k$ ; however, we emphasize the separate effect of  $s$  when strong elastic interaction between sites (e.g. close parallel distributions) renders untenable the self-consistent or effective models for the outside matrix. Equation (3.1) simply says that various viscosities, moduli and geometry of both individual inclusion classes (through the multi-variable function  $f_a$ ) and the overall array of inclusions (through the multi-variable function  $g_a$ ) will affect the characteristic times for continuum element viscoelasticity resulting from individual site relaxation. This is a situation familiar in other relaxing materials (e.g. polymers) where a wide spectrum of characteristic times (e.g. [40]) typically result from a range of micro-mechanisms: there is some advantage in the present situation, however, since the times are actually formally computable when the geometry and moduli of inclusions and matrix are specified.

As an example, we consider the case where inclusions are all roughly ellipsoidal in shape with similar ratios of major to minor axes,  $a/h$ ; the globules are regarded as instantaneously rigid but as having negligible long-term shear-moduli (viz. fluidic in shear), while bulk moduli undergo negligible change in time. The exterior composite, of solid matrix with relaxing fluid inclusions of random orientation, also has a time-dependent effective shear modulus, which appears in the the self-consistent computation of time-dependent moduli, but it will be adequate (for estimates of characteristic times) to use a single average modulus  $L$  which governs shear

deformation of the globule perimeter. If viscosity  $\mu$  applies to each of the inclusion substances, then the characteristic relaxation time must take the form

$$\tau_1 \approx (\mu/L)f_a(a/h). \tag{3.1b}$$

The function  $f_a$  will have dominant contributions from particular powers of  $a/h$ . For instance, if the inclusion is reasonably flat (large  $a/h$ ) then one clearly identifiable source for  $f_a$  is the offset  $\delta \approx \sigma a/L$  (due to shedding of shear stress  $\sigma$ ) between surfaces with  $a$ -axis parallel to shear: since an effective shear-strain  $\gamma \approx \delta/h$  results in the inclusion, it will take time  $\tau \approx \gamma/(\sigma/\mu) \approx \mu a/Lh$  (omitting multiples of order unity) to achieve appreciable relaxation. On the other hand, in inclusions with  $a$ -axis tilted at  $45^\circ$  to shear the stress transfer is somewhat more complex and  $f_a$  may include contributions from  $a^3/h^3$  (especially if the inclusions are rather more tabular than ellipsoidal (Fig. 3b), since appreciable ‘‘squelching’’ flow of fluid must then occur from centre to extremes of inclusion, in redistributing the initial non-uniform elastic stress).

To recognize the differences between  $\hat{G}, \hat{\nu}$  and  $\bar{G}, \bar{\nu}$  let us choose the examples treated in Section (ii), namely a matrix with effective moduli  $G^s, \nu^s$  containing a concentration  $(C - C^s)$  of isotropically distributed flat or spherical inclusions filled with fluid. With reference to eqn (2.18b), we note that spherical pores will cause a shear modulus relaxation by an approximate factor

$$\hat{G}(t = 0)/\bar{G}(t \gg \tau_1 \approx \mu/G^s) \approx \exp \left[ \frac{225(1 - \bar{\nu})^2}{(8 - 10\bar{\nu})(7 - 5\bar{\nu})} (C - C^s) \right]. \tag{3.2a}$$

More exact estimates (analogous to 2.19e) can be obtained by integrating eqn (2.18a) explicitly, after substituting  $\bar{\nu} \approx (3K_m - 2\bar{G})/(6K_m + 2\bar{G})$ , where  $K_m$  is the constant bulk modulus which persists as pores soften in shear only. On the other hand, flat inclusions with very large  $G^*a/hG^s$  (eqn 2.19c) merely ‘‘glue the surfaces together’’ and cause the resulting  $\bar{G}$  to be negligibly different from  $G^s$ ; however, as they relax so does  $\bar{G}$  and the corresponding value of  $\hat{G}/\bar{G}$  can be roughly expressed as

$$(\hat{G} \approx G^s)/\bar{G}(t \gg \tau_1 \approx \mu a/G^s h) \approx \exp[8\alpha(C - C^s)/15\pi]. \tag{3.2b}$$

where  $\alpha$  is of order unity and is deduced from eqn (2.19e) if the site remains effectively isovolumetric. However, we mentioned that there may be a second contribution to relaxation, arising from the redistribution of fluid within an incompletely saturated or multi-faceted site (Fig. 3(a, b)). This further softening may be reasonably well described by the difference between  $\Gamma_E$  in eqn (2.19c) and  $\Gamma_E$  in eqn (2.21a) with  $\Lambda \neq 0$  in  $g_1$  (eqn 2.13b); as a rough estimate, these effects will increase  $\alpha$  by  $1 + 3(1 - 2\bar{\nu})(2 - \bar{\nu})/4[(1 - 2\bar{\nu}) + \Lambda\bar{\nu}(1 - \bar{\nu})]$  for  $t \gg \mu a^3/G^s h^3$ . Of course, the exact estimates of  $\hat{G}/\bar{G}$  are to be extracted from Fig. 2 but eqns (3.2) provide quite accurate mnemonics.

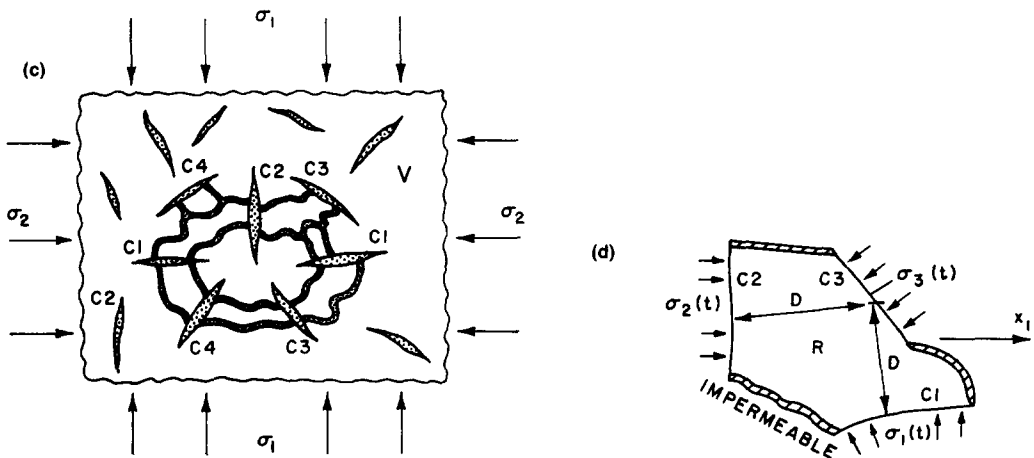


Fig. 3(c). Generic element of isotropically distributed (yet polarized) pore-space with interconnected conduits which may have similar or much smaller characteristic dimensions: pores C1 must drain into others (especially C2) under lower stress. (d). Geometry of isolated sub-domain between flat pores in Fig. 3(c), in the case where they must drain into each other by diffusive fluid transfer through secondary microporosity (and hence give rise to ‘‘modes of consolidation’’); boundary conditions ( $\sigma_3(t)$  = pressure in C3, symmetry-dictated no-flow for ‘‘impermeable’’ regions) are meant only to be suggestive.



(ii) *Local inter-site diffusion of mobile species* ( $\bar{G}, \bar{\nu} \rightarrow G, \nu_u$ ) Figs. 3(b–d).

Although individual sites may relax rather quickly (as given by  $\tau_I$  in the previous section), they will usually not reach equal (thermochemical) potential: for instance, fluid-like inclusions may achieve a pure hydrostatic stress-state but the pressure in them will vary with orientation of any non-spherical globule shape (e.g. sites of type C1 in Fig. 3(c) will sense a stress of order  $\sigma_1$ , while those of orientation C2 sense  $\sigma_2$  approx and C3, C4 etc. have an intermediate induced pressure). Alternately, even spherical globules may not achieve equilibrium by isolated relaxation if a secondary porosity is present interlacing the solid matrix between inclusions. The initial relaxation within sites must be followed by matter transport between sites and thus a second regime of relaxation may be identified, with characteristic time  $\tau_{II}$  (usually  $\gg \tau_I$ ). To ensure the viscoelastic character of the  $\tau_{II}$ -regime (in light of frequent simultaneity with global diffusion, identified later) we shall presently consider an isolated continuum element with sealed boundaries so that only internal (*viz. local*) transport of matter is allowed, albeit via mechanisms which look diffusive!

Again the example chosen is a distribution of viscous globules in a solid matrix (e.g. as shown in Fig. 3c): these have now shed their shear-stress and sustain only a hydrostatic stress dictated by geometry and relative stiffnesses of fluid and surrounding composite. We introduce the possibility of interconnecting fluid ducts, either as secondary matrix porosity or direct contiguity of the sites themselves. In the respective cases, the quantities

$$\tau_{II}^{(1)} \approx D^2/\pi^2 \hat{c}, \quad \hat{c} \equiv K_m \hat{\kappa}/\mu; \quad \tau_{II}^{(2)} \approx a^2/2\pi \hat{c} \quad (3.3a, b)$$

determine the times for transmission between sites or for fluid loss from sites to the secondary porosity (e.g. neglecting compressibility of site fluid). Here  $\hat{c}$  is the diffusivity of the porous surrounding matrix, defined [12] in terms of the permeability  $\hat{\kappa}$ , an effective viscosity  $\mu$  and an appropriate modulus combination of fluid and solid moduli,  $K_m$ ; here, also,  $a$  is the larger representative dimension of the globule and  $D$  is typical distance between globules. The  $\pi^2$  and  $2\pi$  factors come from considering primary consolidation modes between sites (in region  $R$  between sites, schematic Fig. 3d) or from a single site (e.g. Chap. 2 of [13], [41]); thus, there are many other longer characteristic times (for the infinite number of secondary modes) which we have temporarily neglected in considering only the primary part of the relaxation process. Circumstances which allow non-negligible amplitudes of the slower modes should be kept in mind later (when comparing  $\tau_{II}$  to  $\tau_{III}$ ).

Contiguous sites can actually “squirt” fluid directly into each other if they are sufficiently well connected (e.g. [42]): for instance, if the sites are part of a primary porosity network with smallest dimension  $h$  and typical transverse distance  $D$  between sites then a typical equilibrium time can be estimated simply by imposing mass conservation on laminar flow conditions (e.g. Fig. 3b), namely

$$\tau_{II}^{(3)} \approx (\mu/L)f_2(D/h) \quad (3.3c)$$

where  $L$  is again the modulus controlling volume changes of the pores (or sites) as they increase or decrease internal pressure by fluid exchange with adjacent sites (at rate governed by viscosity  $\mu$ ). The function  $f_2$  would be very strong in  $(D/h)^2$  components if long straight capillaries† span between the major pores but an equally relevant model is a direct intersection of misaligned non-equiaxed pores: in the latter case the strongest contribution comes from  $(D/h)^3$  terms while, as before, smaller effects still result from many other powers of  $D/h$  (analogous to the multitude of terms in  $f_1$ , earlier), thereby increasing the possible values of  $\tau_{II}$  for inclusions with very low aspect ratio.

As before, it is essential to have estimates for the order of transition from  $\bar{G}, \bar{\nu}$  to the values  $G, \nu_u$  which apply when all of the sites have achieved equal potential (e.g. fluid pressure). The case of spherical porosity is unique in the sense that  $\bar{G}, \bar{\nu}$  already correspond to essentially unbiased potential distributions between sites (it being impossible to identify a particular site by

†This is one repeatedly used by Biot [14, 15] to lend plausibility to his wave theory. Resulting estimates for critical frequencies thus contain deceptively similar geometric parameters. Indeed, curves for attenuation and dispersion have certain similarity to ours (trivially computed from  $\tau^{(ik)}$  and limiting moduli (e.g. 41)) but the range of relevance must be carefully distinguished.

orientation) and only minor sources of further equilibration appear (e.g. fluid exchange with secondary porosity). Thus the solution of eqn (2.20c) with  $G^*/\bar{G} \approx 0$ , and  $K^*/\bar{K}$  appropriate to the ratio of fluid to solid bulk moduli, provide good estimates of both  $\bar{G}$ ,  $\bar{\nu}$  and  $G$ ,  $\nu_u$ . However, when any ellipticity is present, there is clearly a general need for further flow of fluid, within a representative continuum element, before potential polarization is completely removed. When this has taken place the equal pressures in the various pores is not (except for spherical porosity) even closely determined by the generalised Eshelby-type solutions which would make possible the computations in Section 2.

Fortunately, the moduli  $G$ ,  $\nu_u$  can be computed in terms of the moduli which prevail after all internal pressure increments (over the ambient prevailing values) have dissipated by global diffusion viz. those resulting from solution of eqns (2.21d), or their general ellipsoidal counterparts, with  $G^*/\bar{G} \approx 0 \approx K^*/\bar{K}$ . Thus, whereas intra-site redistribution (such as the "squelching" mechanism, Fig. 3(a), in tabular inclusions) may produce an effective softening of the  $K^*/\bar{K}$  ratio (such that  $\Lambda$  is of order unity), the mechanism of global drainage formally reduces it to absolute zero. We shall compute the resulting "drained" moduli  $G$ ,  $\nu$  in the next section; this second use of  $G$  is not a notational oversight because, in fact, the shear modulus for "undrained" deformation (viz. equilibrated potential between sites, but no fluid escape by global diffusion from a representative continuum element) is easily shown [12] to be the same as that for "drained" deformation (where all internal fluid pressure increments are allowed to extinguish). Further, there is a relation between  $\nu_u$  and  $\nu$ , namely [12]

$$\nu_u = \frac{[3\nu + B(1 - 2\nu)(1 - K/K'_s)]}{[3 - B(1 - 2\nu)(1 - K/K'_s)]}, \quad B \equiv \frac{[1/K - 1/K'_s]}{[V_0/K_f + 1/K - 1/K'_s - V_0/K'_f]}. \quad (3.4)$$

In eqns (3.4),  $K_f$  is the actual bulk modulus of the fluid (which is  $K^*$  only until the pore is allowed to lose fluid),  $V_0$  is the "porosity" or volume concentration of pores,  $K \equiv 2G(1 + \nu)/3(1 - 2\nu)$  is the "drained" bulk modulus and both  $K'_s$  and  $K'_f$  may be approximately identified with  $K^s \equiv 2G^s(1 + \nu^s)/3(1 - 2\nu^s)$ . Clearly, since eqns (2.21a, b) tell us that, in general, both  $\bar{G}$ ,  $\bar{\nu}$  will relax appreciably from  $\bar{G}$ ,  $\bar{\nu}$  to  $G$ ,  $\nu$  and since eqns (3.4) give us an independent relation for  $\nu_u$ , we conclude that substantial alteration in  $\bar{G}$ ,  $\bar{\nu}$  can occur between  $\bar{G}$ ,  $\bar{\nu}$  and  $G$ ,  $\nu_u$ . The amount is to be decided (using eqn 3.4 and Fig. 2, or eqn 3.6 in the next section) for each particular choice of solid/fluid moduli, porosity structure, etc. but a useful observation is that the relaxation is small for spherical porosity and maximum for flat inclusions.

(iii) *Global diffusion between adjacent continuum elements ( $G$ ,  $\nu_u \rightarrow G$ ,  $\nu$ ).*

Even after complete internal equilibration within a representative continuum element (assuming boundaries were sealed off to avoid overlap with the last two regimes), there will still usually be a difference between the resulting thermophysical potentials of the mobile species† in adjacent elements of a body subjected to non-homogeneous stress fields viz. there will be a global (or macroscopic) gradient in potential. As a result, further diffusion will take place on the macroscopic interelement scale and this will eventually lead to a field of stress and species potential which is in equilibrium with the boundary values imposed on these or related variables (e.g. displacement and flow rate). This regime has had central importance in classical consolidation theories of soil and rock mechanics (e.g. [10, 12]) and needs little special consideration here. The central result needed is that the primary "consolidation" time  $\tau_{III}$ , during which is removed most of the imbalance in potential of the mobile species resulting from a stress-field varying with characteristic length  $l$ , can be reasonably estimated [12] by

$$\tau_{III} \approx l^2/4c, \quad c \approx L_m \kappa / \mu \quad (3.5a)$$

where  $\kappa$  is the macroscopic permeability of the overall porous structure,  $\mu$  is fluid viscosity and  $L_m$  is an appropriate modulus of the mixture [12].

For instance, we suppose that stressing is achieved by propagation of a dilational wave through the medium (since, in the linear quasi-static theory, shear produces no diffusion, a

†We deliberately avoid the word "fluid" here as other diffusion phenomena (e.g. impurities in crystalline solid) are to be included where relevant!

feature currently providing guidance in our formulation of a dynamic theory): then the characteristic length  $l$  can be chosen as the wavelength  $\lambda$  and the stress gradient is maintained over this length for a time of order  $\omega^{-1}$  (frequency  $\omega$ ). Appreciable diffusion will take place if we meet the condition

$$\omega^{-1} \geq \lambda^2/4c, \quad \omega \geq c_p^2/4c, \quad \lambda \leq 4c/c_p, \quad c_p \equiv \lambda\omega \quad (3.5b)$$

where  $c_p$  is the bulk wave speed and, for present purposes, can be regarded as effectively determined (to within a time-dependent factor of order 2) for any particular diffusio-viscoelastic medium. Equation (3.5b) is precisely the condition we can derive for appreciable attenuation in the classical Biot theory of wave propagation in porous media: such a recognition leads to better insight on the severe limitations still not being appreciated in physical applications [e.g. 49].†

Some of the difficulty can be seen already by comparing eqns (3.5) and (3.3): if  $l^2/c$  should happen to be of the same order as  $a^2/2\hat{c}$  (or  $D^2/2\hat{c}$ ) then  $\tau_{III}$  would be comparable to  $\tau_{II}$  and, since the artifact of sealed element boundaries is not available (in a continuous body under inhomogeneous stress, for instance), the "viscoelastic" relaxation of Section 3(ii) would be developing simultaneously with the global diffusion of Section 3(iii). This may not cause undue concern‡ unless we remember that both mechanisms have the same character, so that appreciable long-range diffusion may result under global gradients of the polarized site potentials: to be more precise, which measure of potential can we now employ in a law for global diffusion if the sites within a single continuum element are in a state which varies strongly with orientation? The question is considered more carefully in Section 4 but our purpose here is to assess the comparability of  $\tau_{II}$  and  $\tau_{III}$ : we assert that these can be of the same order under many realistic physical conditions and, especially over a geophysical range in which the classical theory of Biot could be argued to have relevance.

In computing  $\hat{c}$  of eqns (3.3) we use (for  $\hat{\kappa}$ ) only the dimensions of the microporosity connecting the major sites (e.g. fluid globules): if the sites are themselves potential passageways for the mobile species then the overall permeability  $\kappa$  may be substantially larger§ (e.g. consider the case of a regular array of pores not quite intersecting but with miniature passageways between them [36]). If as much as a tenfold increase in  $c$  (over  $\hat{c}$ ) can be achieved in this fashion (eqn 3.5) then wavelengths an order of magnitude greater than the maximum pore dimensions will certainly produce  $\tau_{III}$  comparable to  $\tau_{II}$ : although such high-frequency components (approaching microstructural size) may seem irrelevant (because of diffractive scattering [e.g. 43]) they are obviously typical of the wavelengths (eqn 3.5b) needed for appreciable attenuation, by global diffusion, in many representative porous media. However, another particularly interesting situation is that of contiguous sites, when eqn (3.3c) applies with strong contributions from  $(D/h)^3$  terms: since we can correspond  $h^2$  with  $\kappa$ , and thus  $\tau_{II}^{(3)}$  is approximately  $(D^2/c)f_2(D/h)^2$ , we see that  $(l/D)^2$  can be as large as  $D/h$  but  $\tau_{III}$  will still be comparable to  $\tau_{II}$ . Since  $D/h$  can readily be as large as  $10^3$  (viz.  $l/D \approx 30$ ), then wavelengths allowable in the continuum global diffusion theory do involve a simultaneous local equilibration between sites and no single potential is apparent for the purpose of writing an inter-element transport law (like that of D'Arcy or Fick). An excellent example here is the case of highly-fissured rock strata (e.g. [44]).

As we have already mentioned, the values of  $G, \nu$  are to be computed from  $G^s, \nu^s$  by solving eqns (2.20c) and (2.21b) throughout the variation of effective porosity concentration from  $C^s$  to  $C$ , with  $G^*/\bar{G} = 0 = K^*/\bar{K}$ . A reasonable estimate is possible for spherical porosity (if  $\bar{K} \approx K^s$ , i.e. assuming comparable fluid and solid matrix moduli), by noticing that  $\bar{G}$  remains essentially constant at  $\bar{G} = G$ , namely

$$4G \int_{\bar{K}}^{\bar{K}} d\bar{K}/\bar{K}(\bar{K} + 4G) = (C - C^s) = \ln[(4G + \bar{K})K/\bar{K}(4G + K)]. \quad (3.6)$$

†For instance, suppose we choose a typical intact geophysical structure  $c_p \approx 4 \times 10^5$  cm/sec and  $c \leq 10^2$  cm<sup>2</sup>/sec; then we need a wavelength less than  $10^{-3}$  cm for appreciable diffusive attenuation! This will typically be much less than a representative microstructural size for the medium under consideration. Indeed, if the medium is highly fissured, we may well achieve  $c_p \approx 4 \times 10^4$  cm/sec,  $c = 10^4$  cm<sup>2</sup>/sec, i.e.  $\lambda \leq 1.0$  cm but this will, again, be less than microstructural scales.

‡For example, it may appear that the complication is accounted for by viscoelastic moduli in Biot's equations, ignoring the question of a well-defined potential; that fix-up may sometimes be adequate.

§The reader is referred to analogous (somewhat unrealistic) computations of thermal conductivity in a composite [2].

The exponential decay in eqn (2.20b), for  $k = \infty$  and  $\bar{\nu}$  constant, gives a rough idea of the behavior but a better impression is gained from (3.6). However, the actual amount of transition from  $\bar{K}$  to  $K$  is best obtained by first finding  $\bar{K}/K^s$  (for given  $k^s = K^s/K_f$ ,  $\nu^s$ ,  $g = \infty$ ) and then  $K/K^s$  (for  $\nu^s$ ,  $k = \infty$ ,  $g = \infty$ ) from the appropriate curves for spherical porosity in Fig. 2.

Likewise, for flat porosity, the values of  $\bar{K}$ ,  $\bar{G}$  are first extracted from Fig. 2 (using  $\gamma \approx 0$  and appropriate values of  $\nu^s$ ,  $\Lambda$ —probably  $\infty$  for the latter unless the fluid can redistribute and thereby effectively soften); then  $K$ ,  $G$  are obtained (again for  $\nu^s$ ,  $\gamma = 0$ ,  $\Lambda = 0$ ). Unlike the spherical pore case (where  $K \neq \bar{K} \approx \hat{K} \neq K^s$ ,  $G = \bar{G} \neq \hat{G} \neq G^s$ ), we will in general find  $G \neq \bar{G} \neq \hat{G} \neq G^s$  and  $K \neq \bar{K} \approx \hat{K} \approx K^s$ ; by use of eqns (3.4) we then find  $K \neq K_u$  ( $\neq \bar{K}$  either, except for spherical porosity). Thus, there is substantial shear relaxation from  $\hat{G}$  to  $\bar{G}$ , further softening from  $\bar{K}$  to  $K_u$  (and  $\bar{G}$  to  $G$ ) when pores are non-equiaxed and then a final bulk alteration from  $K_u$  to  $K$  as global diffusion is allowed. For the illustrative isotropic model studied, this completes the description of limiting moduli and characteristic times associated with each of the three dominant regimes of time-dependent response.

#### 4. COUPLED CHARACTER OF LOCAL AND GLOBAL DIFFUSION

The foregoing discussion (Sections 2, 3) confirms that there are many circumstances in which local equilibration of fluid potential does not have time to develop fully before appreciable inter-element transfer of fluid (viz. global diffusion) can take place. It is not obvious that any particular tensor-valued potential(s) can then be employed to describe the driving-force for global diffusion; any theory (e.g. [14, 17]) which starts out from that assumption must be applied with great care to the physical situation. Our intention in this section is merely to outline the general character of the local diffusive equilibration process and to suggest some avenues of pursuit toward a tractable theory for dynamic response viz. the extraction of some adequate measure(s) describing the potential in any representative element during the local equilibration process (a topic initiated in Chap. 2 of [13]).

For descriptive convenience we shall limit the discussion to the models (Section 2a) of ellipsoidal fluid inclusions, with appropriate statistical distributions of orientation (e.g. defined by normal  $\mathbf{n}$  to spheroidal inclusions). Suppose the (essentially viscoelastic) short-term shear relaxation (Section 3i) has already taken place and there remains the pressure distributions  $p(\mathbf{n})$ , as determined by eqns (2.2) or their anisotropic analogues (e.g. [21]) with  $G^* \approx 0$ . We first ask if  $p(\mathbf{n})$  has any simple tensor character: toward an answer, we note that the special isotropic eqns (2.2) *et seq.* imply the following form of pressure, before any local flow is allowed

$$p(\mathbf{n}) = \alpha^{(n)} \sigma_{ij}^A n_i n_j + \alpha^{(m)} \sigma_{ij}^A m_i m_j + \alpha^{(t)} \sigma_{ij}^A t_i t_j \quad (4.1)$$

where  $\mathbf{m}$  and  $\mathbf{t}$  are the principal directions (of the ellipsoid) orthogonal to  $\mathbf{n}$ , so that  $\alpha^{(m)} = \alpha^{(t)}$  for the simplest spheroidal case. For any specified component of applied stress  $\sigma_{ij}^A$ , the induced pressure thus corresponds to the same component (in the same coordinate system) of the tensor quantity  $\alpha^{(n)} \mathbf{nn} + \alpha^{(m)} \mathbf{mm} + \alpha^{(t)} \mathbf{tt}$ . Likewise, the pressure in the anisotropic case would be just the appropriate tensor component of a somewhat more extended quantity [21]. We wish to explore the value of this pressure characterization, toward an estimate of the coefficients describing fluid exchange between elements at different levels of applied stress  $\sigma_{ij}^A$ .

An appealing induction from eqn (4.1) is that the fluid pressure among the spheroidal pores ( $\alpha^{(m)} = \alpha^{(t)}$ ) has a general time-dependent behavior of the following kind:

$$p(\mathbf{n}, t) = \int_0^t \{ \dot{S}_{ij}^A(t') [\alpha_A n_i n_j + \alpha_B (m_i m_j + t_i t_j)] + \alpha_C \dot{\sigma}_{kk}^A(t') + \alpha_D \dot{\sigma}^A(t') \} dt', \quad (4.2a)$$

where the evolution scalars  $\alpha_A$ ,  $\alpha_B$ ,  $\alpha_C$ ,  $\alpha_D$  may be functions of the time elapse variable  $t - t'$ ,  $S_{ij} \equiv \sigma_{ij} - \delta_{ij} \sigma_{kk} / 3$  are the deviatoric stress components (in a chosen Cartesian coordinate system), and  $\sigma$  is any other relevant scalar, as yet undetermined. Our separation of the stress into hydrostatic and shear contributions is motivated by the obvious feature that the pore-fluid pressures resulting from  $\sigma_{kk}$  may be already at local equilibrium (i.e.  $\alpha_C$  is independent of  $t - t'$ ), whereas those caused by  $S_{ij}$  must certainly equilibrate by inter-site matter transfer.

However, the individual site-pressures  $p(\mathbf{n}, t)$  must also depend on how the exchange of fluid with adjacent continuum elements (or atmosphere) is controlled: the simplest possible account of this influence is through a single quantity  $m$  expressing the fluid mass content per unit volume. If we continue to limit ourselves to a linear theory for the medium, then we have essentially made the following identifications:

$$\alpha_{A,B,C,D} = \alpha_{A,B,C,D}(t - t'), \quad \sigma \equiv m. \tag{4.2b}$$

Having chosen only a single fluid-mass variable to account for fluid exchange between adjacent representative samples of pore microstructure, there now arises only a single flow vector quantity (components  $q_i$ ), describing that overall fluid transfer but not its composition from discrete inter-site fluxes. The next question concerns the potential which drives the fluid exchange rate  $q_i$ ; since the pore pressures are considered to be responsible for whatever fluxes occur then some measure of  $p(\mathbf{n}, t)$  must be chosen. Inspection of eqn (4.2a) and lack of bias in the discrete transmissivity between sites would quite obviously imply that only the non-polarized components contribute non-cancelling fluxes; a simplistic postulate might then take the form

$$q_i = -\kappa \partial \hat{p} / \partial x_i, \quad \hat{p} \equiv \int_0^t [\alpha_C \dot{\sigma}_{kk} + \alpha_D \dot{m}] dt'. \tag{4.2c}$$

Closure of the theory is now provided by the fluid mass-conservation identity,

$$\frac{\partial}{\partial x_i} \left( \kappa \frac{\partial \hat{p}}{\partial x_i} \right) \equiv \frac{\partial m}{\partial t} = \alpha_D^{-1}(0) \left\{ \frac{\partial}{\partial t} (\hat{p} - \alpha_C(0) \sigma_{kk}) - \int_0^t [\dot{\alpha}_C \dot{\sigma}_{kk} + \dot{\alpha}_D \dot{m}] dt' \right\}. \tag{4.2d}$$

The problem is now reduced to reasonably tractable diffusoviscoelastic response (e.g. [15]); the internal equilibration process is essentially uncoupled from the global diffusion. The stress-strain relations which appear are not difficult to rationalize, when we consider the uniform nature of the pore pressure decay in eqn (4.2a), and we write them as follows:

$$2G\epsilon_{ij} = \sigma_{ij} - \frac{\nu}{1 + \nu} \sigma_{kk} \delta_{ij} - \int_0^t [\dot{S}_{ij}^A (\beta_A \alpha_A + \beta_B \alpha_B) + \delta_{ij} (\beta_C \alpha_C \dot{\sigma}_{kk} + \beta_D \alpha_D \dot{m})] dt'. \tag{4.3a}$$

The constants  $\beta_A, \beta_B, \beta_C, \beta_D$  can be determined quite readily (in accordance with the computations of Section 2) by considering the response immediately after application of a stress increment at  $t = 0$ ; this permits the identification

$$G^{-1} - \bar{G}^{-1} = 2\beta_A \alpha_A(0) + 2\beta_B \alpha_B(0), \quad K^{-1} - \bar{K}^{-1} = 9\beta_C \alpha_C(0). \tag{4.3b}$$

We are still left with two unknown  $\beta$ 's: for those we need to make the reasonable argument that  $\alpha_A(t)/\alpha_B(t)$  and  $\alpha_C(t)/\alpha_D(t)$  are constant, so that  $\beta_B = \beta_A$  and  $\beta_D = \beta_C$  without loss of generality. Since  $\alpha_A(0) \approx \alpha^{(n)}$  and  $\alpha_B(0) \approx \alpha^{(m)}$  can readily be determined from the computations performed in Section 2, while  $\alpha_C(0)$  and  $\alpha_D(0)$  are adequately rationalized elsewhere [12], there remains only the problem of establishing the normalized character of the  $\alpha(t - t')$  functions, especially  $\alpha_{A,B}$  (since  $\alpha_{C,D}$  will be reasonably constant). Our discussion of this latter question should help to emphasize how special the assumptions are which led to eqn (4.2).

Let us temporarily restrict our attention to local diffusion; we can regard the variously oriented sites as a bank of interconnected reservoirs, exchanging fluid in accordance with some transmissivity function  $k(\mathbf{n}', \mathbf{n})$  which defines the rate at which fluid mass flows from sites with orientation  $\mathbf{n}'$  to those with normal  $\mathbf{n}$  under a unit difference of pressure between the sites. As a result, any particular site ( $\mathbf{n}$ ), receiving fluid from all the others (denoted generically by  $\mathbf{n}'$ ), changes its mass fluid content: if the fluid has bulk modulus  $K_f$  and the modulus controlling matrix deformation (under increments of internal pressure) is  $E^e$ , then a formal statement of the

fluid exchange conditions is, with reference fluid density  $\rho_0$

$$M(\mathbf{n})\frac{\partial}{\partial t}p(\mathbf{n}) = \int k(\mathbf{n}', \mathbf{n})[p(\mathbf{n}') - p(\mathbf{n})] dS(\mathbf{n}'), \quad M \equiv \rho_0 V \left( \frac{e}{E^e} - \frac{1}{K_f} \right) \quad (4.4)$$

where the integral is taken over all possible values of  $\mathbf{n}'$  (viz. the surface of a unit sphere), and the ellipticity ratio  $e$  is needed to define the amount by which the effective site volume  $V(\mathbf{n})$  changes under increments of internal pressure (thus  $e \approx 1.0$  for spherical sites,  $e \approx a_1/a_3$  for flat crack-like inclusions). For the present phenomenological discussion, we neglect the fact that  $E^e$  will be somewhat time-dependent.

We attempt a solution of the following form, with amplitudes  $\hat{q}_j$ ,

$$p(\mathbf{n}, t) = p_0 + \sum_{j=1}^{\infty} \hat{q}_j p_j(\mathbf{n}) \exp\{-\omega_j t\} \quad (4.5)$$

as motivated by the analogy with interconnected reservoirs (or electrical circuits of capacitors and resistances). This will solve eqn (4.4) if the "modes"  $p_k$  and associated decay times  $\omega_k^{-1}$  satisfy the integral equation

$$(K(\mathbf{n}) - \omega_j M(\mathbf{n}))p_j(\mathbf{n}) = \int k(\mathbf{n}', \mathbf{n})p_j(\mathbf{n}') dS(\mathbf{n}') \quad (4.6)$$

where  $K(\mathbf{n})$  is the normalized integral of  $k(\mathbf{n}', \mathbf{n})$  over all possible orientations  $\mathbf{n}'$ . The eigenvalue problem (e.g. [45]) implicit in eqn (4.6), is substantially more intelligible as the equivalent matrix problem, in which a finite proportion of the site volume is assigned to each of a finite group of orientations. The latter view converts eqn (4.6) to an  $N \times N$ -matrix equation on the  $N$  distinct values of any pressure mode  $\{p\}$ , namely

$$[(K - \omega M(\mathbf{n}_i))\delta_{ij} - k(\mathbf{n}_i, \mathbf{n}_j)]\{p\} = \{0\}, \quad i, j = 1, \dots, N. \quad (4.7)$$

An example is  $k(\mathbf{n}_i, \mathbf{n}_j) = K/N$  and constant  $M$ ; then the non-trivial eigenvalues of eqn (4.7) are degenerate,  $\omega_k = K/M$  (analogous to our estimates in Section 3), and it is obviously possible to satisfy initial conditions (eqn 4.1) since a restriction on shapes of  $\{p\}$  arises only in the respect of zero average, if we employ (4.5). An important consequence of eqns (4.6) and (4.7) is that there will actually be (an infinity of) values of  $\omega_k$  substantially lower than  $K/M$  in the usual physical circumstances where  $k(\mathbf{n}', \mathbf{n})$  is not a constant; these imply longer relaxation times and cannot be ignored because a single mode cannot be expected to satisfy initial conditions of the kind in eqn (4.1). Thus, our estimates for  $\tau_{11}$  in Section 3 are the lowest possible because they assume that all sites exchange fluid as if they were adjacent, whereas the permeability  $k$  between site  $\mathbf{n}'$  and site  $\mathbf{n}$  must actually be computed as the inverse of resistances summed over the multiple-spacing distances that must (by simple probability considerations) separate the two sites; sites of all orientations cannot be packed together with only a single representative spacing separating any two of them.

A number of suggestions may now be made about valid measures of potential for global diffusion. In particular, eqns (4.1) and (4.2) may motivate the extraction of an orientation average; our results for uniform  $k(\mathbf{n}', \mathbf{n})$  support this idea because they imply that  $p_0$  in eqn (4.5) is indeed precisely that average so that it certainly becomes the fluid potential after sufficient time has elapsed. However, we must in general expect that the fluid pressure average will not be constant in the local equilibration regime (even if  $\alpha_C, \alpha_D$  are constant in eqn 4.2d). Indeed, the presence of an average like  $p_0$  (constant during the local equilibration process) does strongly appeal for its use to govern inter-element fluid motion and suitable behavior of  $k(\mathbf{n}_i, \mathbf{n}_j)$  may permit the initial conditions (eqn 4.1) to be satisfied by a modal response which allows such a suitable potential  $p_0$ ; however, most physical circumstances will not be amenable to such a simple treatment and much better understanding of representative microstructures is being pursued (as motivated by efforts in [36] and internal variable concepts in [13]).

##### 5. IMPLICATIONS FOR DYNAMIC INELASTIC BEHAVIOR AND CONCLUSION

In all that preceded we have assumed that the initial geometry of the microstructure is preserved: thus, it is feasible for inclusions (especially pores) to revert back to their original shapes when the stress increment, causing deformation and fluid flow, is removed. This

elasticity may be provided by the coherence of the surrounding matrix and the assumption that inclusion material (e.g. fluid) stays in contact with the interface which it forms with the matrix material (hence our choice of the word “saturated” (e.g. [12])). However, the stress may indeed be sufficient to cause either irrecoverable deformation in the solid matrix or actual micro-structural alteration (such as growth of crack-like porosity or grain-boundary sliding). The former may be simply (for instance) just crystalline dislocation motion, dictated by the local (micro) resolved shear stress on available slip planes but an important example of the latter is brittle growth of crack-like porosity. We wish to make a few brief observations on the relevance of our preceding discussions of inclusion fluid pressure and locally non-equilibrated potentials in deciding on inelastic constitutive formulations for materials incorporating these sources of “plasticity”.

In the first place, it is fairly obvious that the presence of an equilibrated fluid pressure ( $p$ ) in inclusions, along with an equal (i.e. negative compressive) exterior hydrostatic stress  $\sigma_{ij}^{\wedge} \approx -p\delta_{ij}$  gives rise to a micro-stress distribution which is essentially uniform and non-deviatoric (except in the presence of any active micro-heterogeneities which have not been treatable as controllable inclusions sites viz. have been lumped into the region called solid matrix). Thus, it is only the other part of the exterior stress, defined by  $\sigma_{ij}^{\wedge} + p\sigma_{ij}$ , which is dominantly operative in producing the micro-stresses needed for inelastic deformation: this is just another way of arriving at an approximate vindication (e.g. Chap 2 of [13]) of the classical “effective stress” law of soil mechanics (e.g. [20]). It becomes somewhat exact when the irreversibility is due to propagation of pores (or inclusions) which are sufficiently flat that there is a severe stress concentration on their perimeter whenever interior stress is not equal to the exterior applied stress on the (perhaps composite) element containing this crack-like porosity. It seems, then, that one should just test the inelastic behavior of the material, allowing ample time for fluid pressure dissipation through the walls to the surrounding atmosphere: if  $f(\sigma_{ij})$  describes some inelastic characteristic of the material (e.g. a “yield” function) in these tests, then  $f(\sigma_{ij} + p\delta_{ij})$  should be the appropriate function when  $p$  is determined by separate means in solution of a field problem with non-homogeneous stress conditions.

Since an effective stress law of this kind (even with extensions like  $\sigma_{ij}^{\wedge} + \xi p\delta_{ij}$ ,  $0 < \xi \leq 1.0$ , e.g. in attempting to account for non-acute asperity contact situations) has played such a central role in quasi-static analyses of soil and rock mechanics (e.g. [46, 47]), it is now opportune to ask how well it can be extended (as often assumed) to the dynamic context. The first complication arises in defining  $p$  for stressing times shorter than  $\tau_{II}$  (Section 3): again, the orientation average may be proposed (as we rationalized somewhat, for the potential governing global diffusion, in Section 4). Indeed, experiments may be conducted in which induced pore-pressure is measured by piezometer for a dynamically stressed element: we see that such a measurement is ambiguous for flat pores and use of it or extension of quasi-static criteria to dynamic formulations warrants great care. The question then remains as to whether the relation between inelastic behavior and apparent fluid pressure, observed in these experiments, has any determinable connection to the actual mechanisms causing failure or can even be linked to corresponding observations under sufficiently slow stressing conditions.

That the actual local pressures in fluid inclusions can be more important (than any average measure of fluid stress) for the actual mechanisms causing failure is seen quite readily by example. If a flat inclusion can extend its surface area whenever stresses on the perimeter exceed some toleration criterion (e.g. like energy release rate for crack-like propagation), then such propagation depends mainly on the difference between pressure in the inclusion and the normal component (of the exterior stress field) on the surface. This particular inclusion could thus extend irreversibly (even to the extent of linking up with other pores and causing a macroscopic fracture), even while the overall average of fluid pressure (e.g. zero for external shear stress) might suggest that the effective stress combination is not adequate for inelastic deformation. Likewise, the micro-stress field in the matrix around such an inclusion depends predominantly on a combination of the applied exterior stress and the individual fluid pressure (or even shear-stress, for  $t < \tau_I$ ) in that site: occurrence of matrix inelasticity (like plastic slip) will, thus, not be determined by any single average of the “pore-pressures” but rather by extremes and details of the inclusion-pressure distribution. Averages will be helpful only insofar as they allow monitoring of such extremes: thus an effective stress law (based on  $\sigma_{ij}^{\wedge} + \xi p\delta_{ij}$ ) can be considered relevant to rapid stressing regimes only insofar as  $\xi$  (for instance)

can be chosen to represent the most critical individual inclusion pressure, as related to the average "pore-pressure"  $p$ .

In conclusion, then, we may identify the problems which will merit attention in subsequent papers: (1) classical and mixture theories, even for low-amplitude wave propagation in porous media, must be confined to the limited realm of their physical applicability (e.g. as dictated by criteria like eqns 3.5); an improved formulation must be found, based on the physical considerations in this paper, to isolate the viscoelastic (e.g. [19, 48]) from the diffusive modifications of stress waves, especially in light of continuing inability to experimentally identify twin wave-types predicted by the theory employed (e.g. [49]). As well (2) it will be essential to provide more realistic descriptions of response to shock waves in many practical endeavors (e.g. blasting); the micro-structure pertaining in the vicinity of a narrow shock front is undoubtedly more complicated than the common hydrodynamic assumptions (e.g. [17]) and the non-equilibration of fluid state to a single pore-pressure, as described here, merits primary attention. Lastly, we might mention the value of (3) further computations for the limiting elastic responses in the case of overall anisotropic behavior of the porous composite (viz. extensions of Section 2); some of these have been attempted (e.g. see review in [21]), and indeed the isotropic distribution of non-equiaxed sites presents the most extreme example of polarised pore-fluid potential, but a complete set of physically consistent results (analogous to Fig. 2) would certainly be worth having for practically important assumptions like transverse isotropy.

*Acknowledgements*—This work was supported by the National Science Foundation, under Grant NO. ENG76-82007 at Massachusetts Institute of Technology. I am grateful to G. Dvorak, J. Hutchinson, J. Walsh, one anonymous reviewer and especially J. R. Rice for supplying various helpful pieces of information.

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